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Evaluation of cascaded inertial vibration isolation systems

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EVALUATION OF CASCADED INERTIAL

VIBRATION ISOLATION SYSTEMS

By

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A

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ABSTRACT

Cascaded inertial vibration isolation systems are examined in this report. Systems employing one, two or three masses in series on isolators are investigated. The objective is to determine if the cascaded systems have appreciable advantages over the classical single mass system.

The equations of motion for these systems are derived by applying Newton's second law of motion. The homogeneous and steady state sinusoidal excitation solutions have been established. Transmissibility of forces and moments to the foundation has been obtained for several cases of force excitation. Comparisons of the cases investigated are based upon the principal mode frequencies, mode shapes, center of mass displacements and transmissibilities.

The ratio of the maximum forcing function to the total weight of the system has in all cases been held at a level of one to four. The spring coefficients have been chosen such that each sub-system, i.e., one mass and its direct supporting springs, have a natural frequency of approximately one cps.

The amplitudes of the top mass, in general, increase with the natural frequency bandwidths. Cases which do not follow this trend are those where some isolators connect the top mass directly to the foundation and cases having a lighter mass at the top. Transmissibilities are the lowest for the three mass system and the highest for the conventional one mass system.

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NOMENCLATURE

$[A]$	= Eigenvector modal matrix
$[A]^T$	= Transpose of the eigenvector modal matrix
$\lceil a_n \rfloor$	= Normalizing constants; diagonal matrix
C-	= Cascade system number
c.g.	= Center of gravity
cps	= Cycles per second
Eccx	= Eccentricity of $\tilde{F}$ from the reference axes
Eccy	= Eccentricity of F from the reference axes
F	= Forcing function in the X direction
$\tilde{F}$	= Forcing function in the Y direction
F_{ij}	= Force transmitted to the foundation in the X direction at mass i from isolator j
$\tilde{F}_{ij}$	= Force transmitted to the foundation in the Y direction at mass i from isolator j
h_i	= Half the height of M_i $(i = 1,2,3)$
h_{ij}	= Distance of isolator from XZ plane
I_i	= Mass moment of inertia of M_i
j	= Subscript indicating the number of isolators (same for the X and Y directions)
$[\bar{K}]$	= System stiffness matrix after coordinate transformation
$[K_i]$	= Stiffness matrix
$K_{\ell m}$	= The ℓm element of a stiffness matrix
k_{ij}	= Isolator stiffness in the X direction
$\tilde{k}_{ij}$	= Isolator stiffness in the Y direction
k_{1cs}	= Stiffness of isolator connecting M_1 and M_2

k_{2cs} = Stiffness of isolator connecting M_2 and M_3

k_{3cs} = Stiffness of isolator connecting M_1 and M_3

ℓ_i = Half the length of M_i

ℓ_{ij} = Distance of isolator from YZ plane

$\overline{m}$ = A normalization constant

M_o = Moment applied to the top mass in the **Z** direction

M_T = Total moment transmitted to the foundation

M_i = Mass of the ith rigid body

T_{XX} = Force transmissibility in the X direction due to a forcing function in the X direction

T_{MX} = Moment transmissibility due to a forcing function in the X direction

T_{YY} = Force transmissibility in the Y direction due to a forcing function in the Y direction

T_{XY} = Force transmissibility in the X direction due to a forcing function in the Y direction

T_{MY} = Moment transmissibility due to a forcing function in the Y direction

X_{ij} = Displacement of the jth isolator in the X direction

Y_{ij} = Displacement of the jth isolator in the Y direction

X_i, Y_i, θ_i = Displacements of the c.g. of M_i

$\ddot{X}_i, \ddot{Y}_i, \ddot{\theta}_i$ = Accelerations of the c.g. of M_i

$\{\overline{\eta}\}, \{\xi\}$ = Transformed coordinates

ω = Natural frequency

Ω = Forcing frequency

$[\Phi]$ = Normalized modal matrix

$[\Phi]^T$ = Transpose of the normalized modal matrix

CHAPTER I

INTRODUCTION

There are two primary aspects to the problem of vibration isolation: first, the isolation of unbalanced forces created by rotating and reciprocating machinery such as fans, compressors, electric motors and diesel engines; and second, the attenuation of base motion which occurs in aircraft, ships and similar automotive vehicles. The principal objective in the first mentioned aspect is the reduction in the magnitude of the force transmitted to the support of the machinery. In the second aspect, the principal objective is a reduction in vibration amplitude imposed on the mounted equipment.

For vibration isolation, in general, it is customary to mount the equipment directly upon spring isolators or upon a rigid body which is supported on isolators. It is also possible to use two or more rigid bodies on spring isolators in series, thus forming what is termed cascaded inertial mass systems (see Appendix B and Figures 2.5 and 2.8).

In industrial applications where spring isolators or rigid bodies supported on isolators are used as machine vibration mounts, the fundamental requirement is to keep the natural frequency of the mass-spring system substantially below the disturbing frequency. In practice, the equipment itself is many times the cause of vibration as it may have unbalanced forces occurring in it. Likewise, in other cases the equipment may be a sensitive device which needs to be isolated from a vibration type environment. In general, the inertial mass type of isolation system is used when very low force transmissibility factors are desired or when very small fragile instruments are to be isolated from the motions of the environment.

The classical theory of inertial mass vibration isolation [1] has employed, in general, only the one mass system. The main purpose of this study is to consider two or more masses to determine if better systems can be designed using available isolators. Their behavior in isolating vibrations is investigated and compared with that of the classical one mass system. Sixteen different cases employing multiple masses have been considered by changing the combinations of masses and isolators.

The isolators commonly used in practice on isolation masses are air springs [7], as air is an ideal load carrying medium. These springs have rubber which is highly elastic, the spring rates can be easily varied, and they are not subjected to permanent set. Air springs also have certain additional advantages over conventional springs. Their spring rate can be changed over a small range by changing internal pressure, their natural frequency can be kept constant by adjusting the pressure in active systems and their load-deflection curve can be adjusted. In this study, however, only systems with passive rubber air springs are considered.

A. Contents of Thesis

Chapter II contains the assumptions and conditions of an inertial mass isolation system. Also the equations of motion are derived. The number of equations obtained have been simplified by considering the rigid body to be symmetrical about one plane. In general practice, this condition is desirable to eliminate as much coupling of displacements in different planes as possible.

One, two and three mass systems are described and their governing differential equations of motion derived. These equations have been derived by applying Newton's second law of motion. The differential equations for each case have been put into general matrix form such that the solutions to the general matrix form can be obtained and applied to all cases of interest.

In Chapter III the solutions to the general matrix differential equation are obtained. Firstly, the homogeneous solution has been obtained. This was done by formulating the standard eigenvalue problem from the matrix differential equation and obtaining numerically, by use of a digital computer, the eigenvectors and the eigenvalues. Secondly, the forced excitation solution for applied sinusoidal forces and moments has been found using the classical superposition of normal modes approach. The forced excitation solution gives the time dependent solution for the c.g. displacements. Total forces and moments transmitted to the foundation due to the c.g. displacements have been established and then the maximum forces and moments transmitted have been established for the steady state. Force and Moment transmissibilities have been defined and plotted in curve form for several different cascaded systems.

One case of a three mass system has been considered to verify the equations of motion and the computer program written for the solutions of the cascaded systems. For this case, the isolators have been placed symmetrically about the reference axes. The stiffness coefficients of the isolators connecting the top two masses have been made much stiffer than the stiffness of the isolators connecting the third mass and the foundation. Thus,in the limiting case the degrees of freedom of the system have been reduced. Consequently, the number of equations governing the motion of the limiting case has been reduced. Also the computer program written for the cascaded systems can be verified as the solutions to the limiting case can be calculated by hand and can be compared with the computer program results.

In Chapter IV comparisons of eighteen different cascaded systems have been made based upon the principal mode frequencies and mode shapes. For each case the frequency bandwidth, i.e., the difference between the largest and the smallest natural frequency root has been compared. The mode shapes were examined for any major differences in the relative displacements. Also, comparison of all the cases under forced excitation has been made based on the amplitudes of the top mass and the force and moment transmissibilities. The amplitudes and transmissibilities for each case have been obtained for varying forcing frequencies from three to fifteen cycles per second.

The major emphasis of comparison is based on systems from each category of one, two or three masses which have the largest and the smallest natural frequency bandwidth. Because the system response is inversely proportional to the difference of the principal mode frequencies squared and the forcing frequency squared, the frequency bandwidth comparisons were made to determine if bandwidth is a criterion by which systems of this type could be evaluated.

CHAPTER II

CASCADED INERTIAL VIBRATION ISOLATION SYSTEMS

A. Assumptions and Conditions

The assumptions used to consider the inertial mass isolation system are listed below. A cascaded system is defined to be one composed of two or more rigid body masses interconnected by springs.

(a) Inertial masses are assumed to be rigid bodies (in the frequency range of interest) mounted on multiple isolators.

(b) Isolators at each point are linear, massless springs and are allowed to have stiffness in both the X and Y directions.

(c) Each rigid body has, in general, six degrees of freedom. By introducing simplifying approximations for practical problems, decoupling of certain modes of vibrations can be achieved. This is done to make the analysis of the problem easier. The extent of the simplification depends on the degree of symmetry involved. In this report, the body and the isolators are assumed symmetrical in one plane, i.e., the YX plane (see Figure 2.1). Therefore the body will have only three degrees of freedom and its motion will consist of three coupled modes of vibration each having vertical, horizontal and rotational components.

(d) The input forces are assumed to be along the line of symmetry, i.e., in the YX plane. Any coupling out of the XY plane leads to negligible displacements in other planes.

(e) The reference axes are selected at the c.g. of the body and coincide with the principal inertial axes of the body.

(f) Forced response is considered only for steady state sinusoidal conditions.

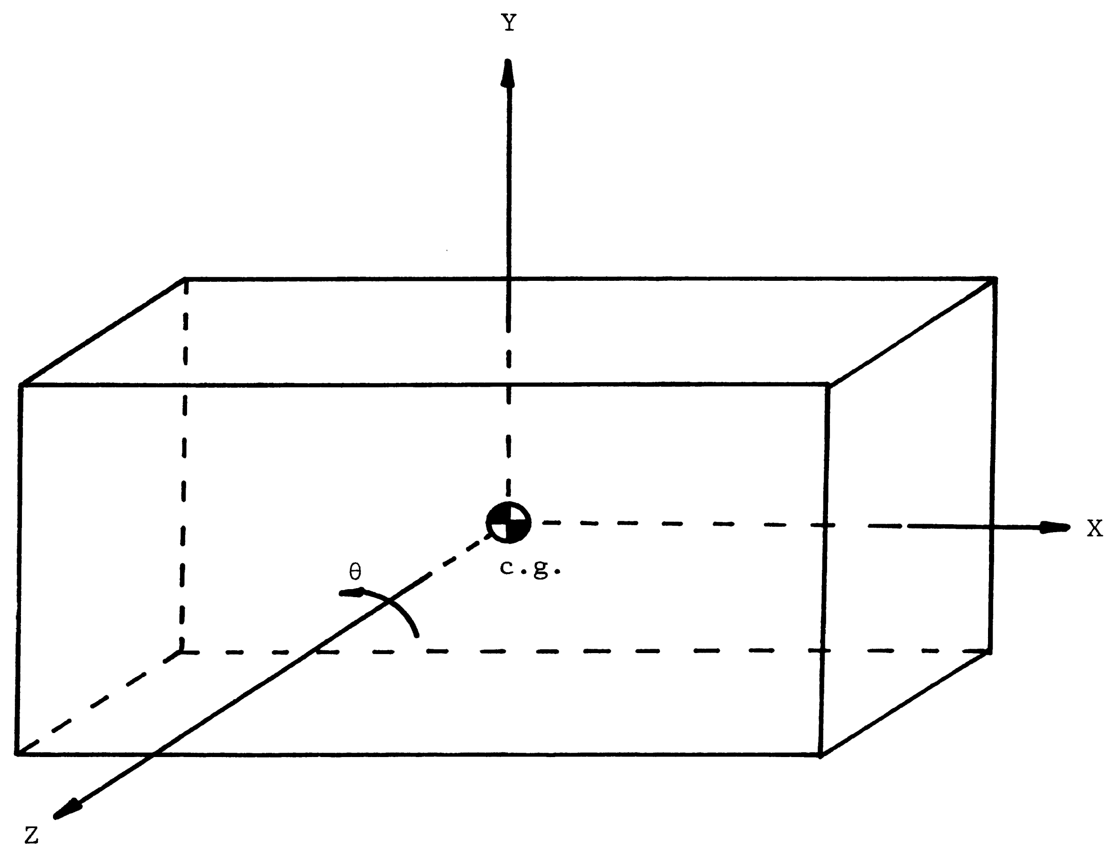


Fig 2.1 Isometric View of a Rigid Body

B. One Mass System and Governing Differential Equations

Figure 2.2 shows a rigid body of mass M_1 on isolators. This represents the classical one mass inertial isolation system. The body is acted upon by external forces F, $\tilde{F}$ and a moment M_o which are sinusoidal. The forces represent the reaction of a vibration machine or any other type of unbalanced machine which might be mounted on the mass. To apply Newton's second law consider the body to experience translational displacements X_1 and Y_1 at its center of gravity and a rotational displacement θ_1 about an axis through the c.g. as shown in Figure 2.4. Because of displacements X_1, Y_1 and θ_1 forces in the isolators in the X and Y directions are created. Figure 2.3 shows a free body diagram of the rigid body showing all of the spring forces acting on it.

According to Newton's second law:

$$\overline{f} = m\overline{a}$$

where: $\overline{f}$ = net force vector acting on the body,

m = mass, and

$\overline{a}$ = acceleration vector of the body.

And,

$$\overline{T} = I\overline{\alpha}$$

where: $\overline{T}$ = torque vector about the c.g.,

I = mass moment of inertia, and

$\overline{\alpha}$ = rotational acceleration vector.

The sign convention for displacements, forces and moments is shown in Figure 2.5.

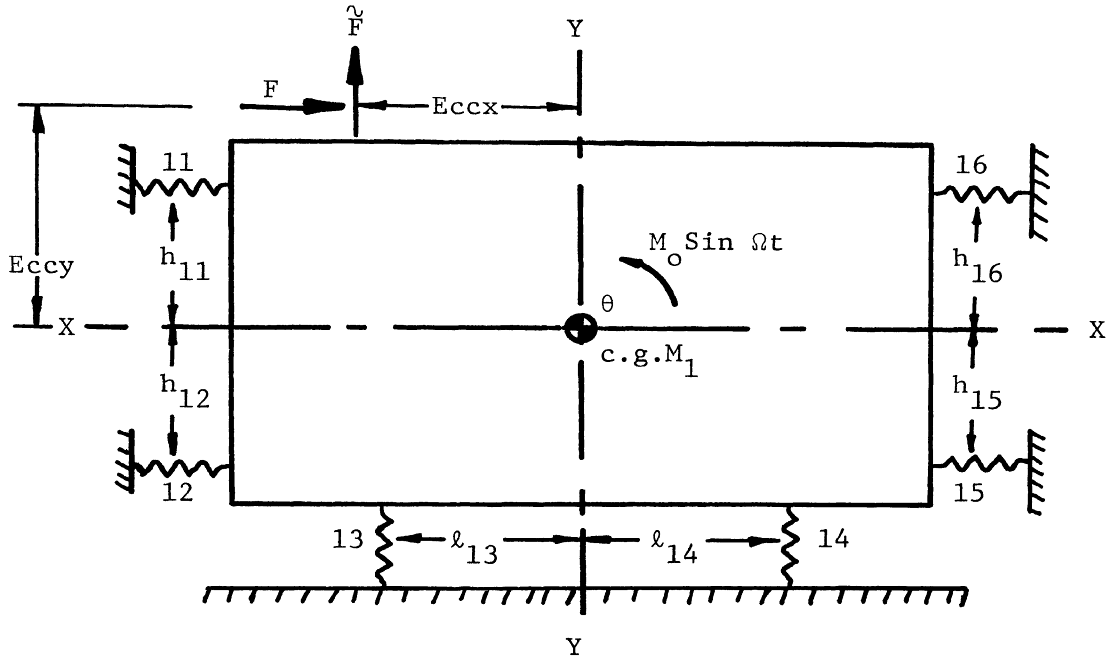


Fig. 2.2 One Mass System

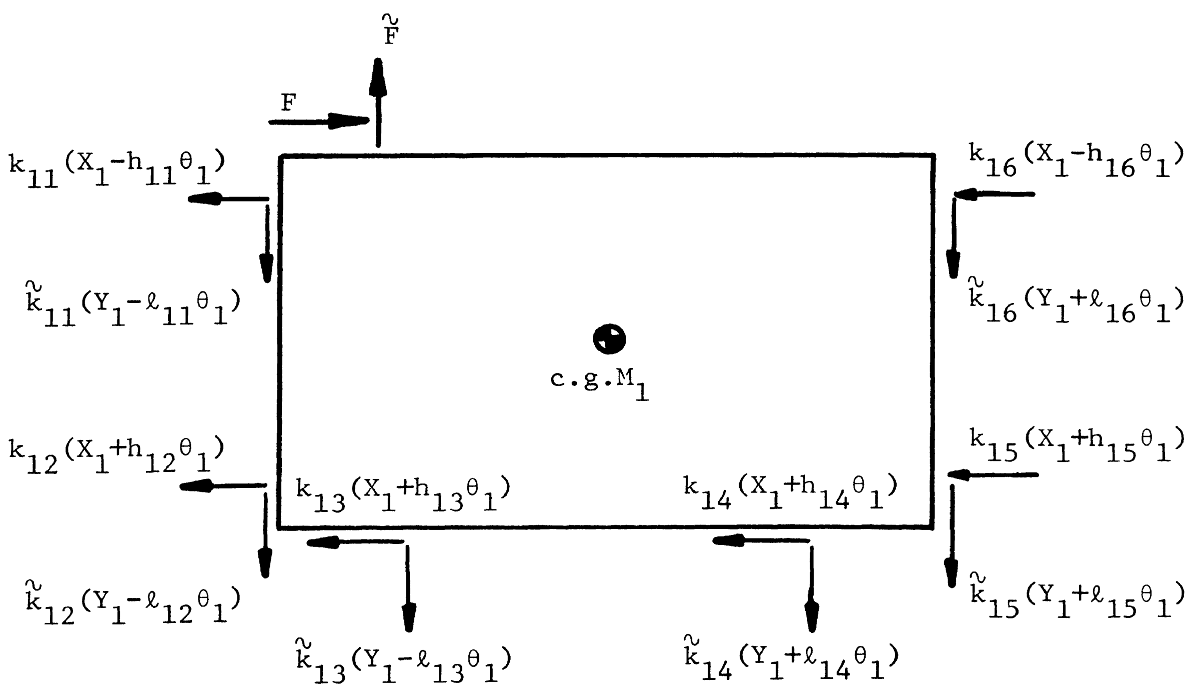


Fig. 2.3 Free Body Diagram of M_1

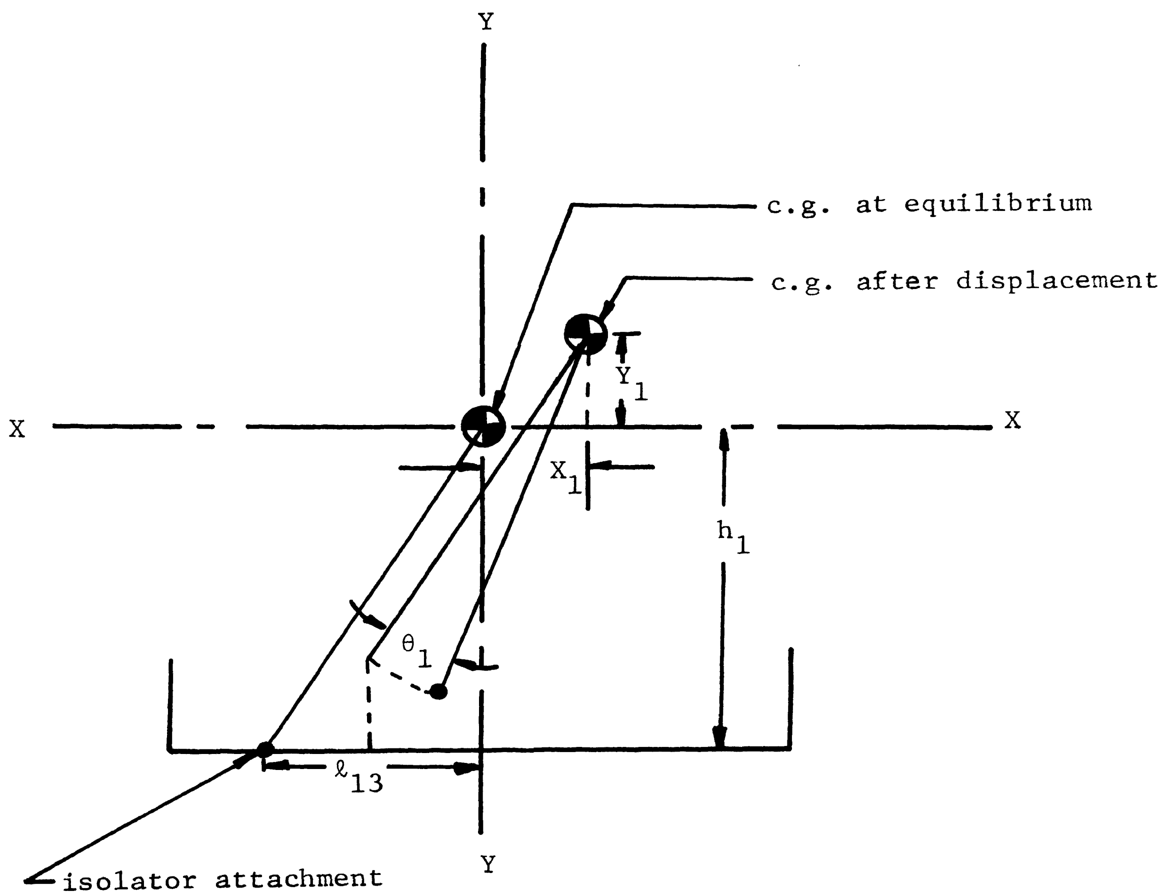


Fig. 2.4 Geometry of the c.g. Displacements

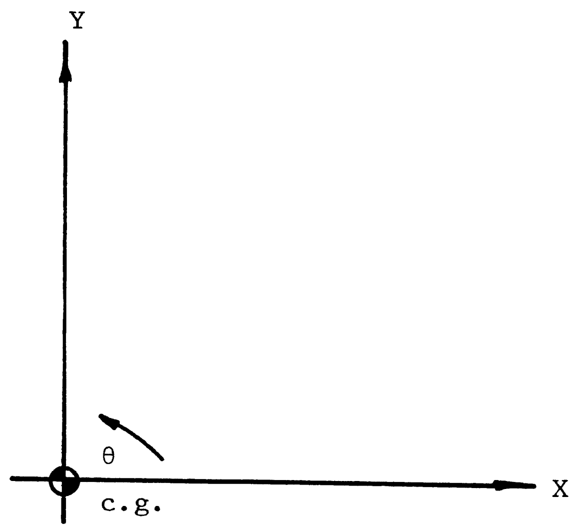


Fig. 2.5 Sign Convention for Displacements, Forces, and Moments

The coordinate axes are located at the c.g. of the mass. The distances of the isolators from the coordinate axes are positive if taken upwards or towards the right and negative if taken downwards or towards the left. Applying Newton's second law, the differential equations of motion considering linear approximations and small displacements are:

$$M_1\ddot{X}_1 = - k_{11}(X_1-h_{11}\theta_1) - k_{12}(X_1+h_{12}\theta_1) - k_{13}(X_1+h_{13}\theta_1) - k_{14}(X_1+h_{14}\theta_1) - k_{15}(X_1+h_{15}\theta_1) - k_{16}(X_1-h_{16}\theta_1) + F \sin \Omega t.$$

$$M_1\ddot{Y}_1 = - \tilde{k}_{11}(Y_1-\ell_{11}\theta_1) - \tilde{k}_{12}(Y_1-\ell_{12}\theta_1) - \tilde{k}_{13}(Y_1-\ell_{13}\theta_1) - \tilde{k}_{14}(Y_1+\ell_{14}\theta_1) - \tilde{k}_{15}(Y_1+\ell_{15}\theta_1) - \tilde{k}_{16}(Y_1+\ell_{16}\theta_1) + \tilde{F} \sin \Omega t.$$

$$
\begin{aligned}
I_1\ddot{\theta}_1 = \; & \tilde{k}_{11}\ell_{11}(Y_1-\ell_{11}\theta_1) + \tilde{k}_{12}\ell_{12}(Y_1-\ell_{12}\theta_1) + \tilde{k}_{13}\ell_{13}(Y_1-\ell_{13}\theta_1) \\
& - \tilde{k}_{14}\ell_{14}(Y_1+\ell_{14}\theta_1) - \tilde{k}_{15}\ell_{15}(Y_1+\ell_{15}\theta_1) - \tilde{k}_{16}\ell_{16}(Y_1+\ell_{16}\theta_1) \\
& + k_{11}h_{11}(X_1-h_{11}\theta_1) - k_{12}h_{12}(X_1+h_{12}\theta_1) - k_{13}h_{13}(X_1+h_{13}\theta_1) \\
& - k_{14}h_{14}(X_1+h_{14}\theta_1) - k_{15}h_{15}(X_1+h_{15}\theta_1) + k_{16}h_{16}(X_1-h_{16}\theta_1) \\
& + M_o \sin \Omega t - (\tilde{F}\ \mathrm{Eccx} + F\ \mathrm{Eccy}) \sin \Omega t.
\end{aligned}
$$

Simplifying, the equations are:

$$
\begin{aligned}
& M_1\ddot{X}_1 + (\sum_j k_{1j})X_1 - (\sum_j k_{1j}h_{1j})\theta_1 = F \sin \Omega t \\
& M_1\ddot{Y}_1 + (\sum_j \tilde{k}_{1j})Y_1 + (\sum_j \tilde{k}_{1j}\ell_{1j})\theta_1 = \tilde{F} \sin \Omega t \\
& I_1\ddot{\theta}_1 - (\sum_j k_{1j}h_{1j})X_1 + (\sum_j \tilde{k}_{1j}\ell_{1j})Y_1 + \\
& (\sum_j k_{1j}h_{1j}^2 + \sum_j \tilde{k}_{1j}\ell_{1j}^2)\theta_1 = \tilde{M}_o \sin \Omega t
\end{aligned}
$$

where: j = number of isolators 1,2,3,....,n.

Writing these three equations in general matrix form:

$$[M_1]\{\ddot{\eta}\} + [K_1]\{\eta\} = \{F\} \tag{2.1}$$

where:

$$
[M_1] = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & I_1 \end{bmatrix},
$$

$$
\{\ddot{\eta}\} = \begin{Bmatrix} \ddot{X}_1 \\ \ddot{Y}_1 \\ \ddot{\theta}_1 \end{Bmatrix}, \quad \{\eta\} = \begin{Bmatrix} X_1 \\ Y_1 \\ \theta_1 \end{Bmatrix},
$$

Note that $\tilde{M}_o \sin \Omega t = (M_o - \tilde{F}\ \mathrm{Eccx} - F\ \mathrm{Eccy}) \sin \Omega t$ for all cases.

$$[K_1] = \begin{bmatrix} \sum_j k_{1j} & 0 & -\sum_j k_{1j} h_{1j} \\ 0 & \sum_j \tilde{k}_{1j} & \sum_j \tilde{k}_{1j} \ell_{1j} \\ -\sum_j k_{1j} h_{1j} & \sum_j \tilde{k}_{1j} \ell_{1j} & \sum_j \tilde{k}_{1j} \ell_{1j}^2 + \sum_j k_{1j} h_{1j}^2 \end{bmatrix}, \text{ and}$$

$$\{F\} = \begin{Bmatrix} F \sin \Omega t \\ \tilde{F} \sin \Omega t \\ \tilde{M}_o \sin \Omega t \end{Bmatrix} .$$

Note that the mass matrix $[M_1]$ is always a diagonal matrix and the stiffness matrix $[K_1]$ a symmetrical matrix.

C. Two Mass System and Governing Differential Equations

Figure 2.6 shows a possible two mass inertial isolation system. In application, the machine or instrument to be isolated would be mounted on mass M_1. Here M_1 is considered to have the same generalized displacements as in the one mass system while the c.g. of M_2 is to have displacements X_2, Y_2 and θ_2. Also, the assumption is made that the displacements of M_1 are larger than those of M_2. This assumption is made only for convenience in writing the equations of motion. The relationship between the displacements is governed by the equations for all time "t" once the equations are formed.

The external forces and moments which would be imposed act on M_1 only. This occurs because the equipment to be isolated or whose unbalanced forces are to be attenuated is always on the top mass.

The sign convention for the distances of isolators from the coordinate axes is the same as in the one mass system. In each mass, the sign convention used for displacements, forces and moments is as shown in Figure 2.7. The coordinate axes are placed at the center of

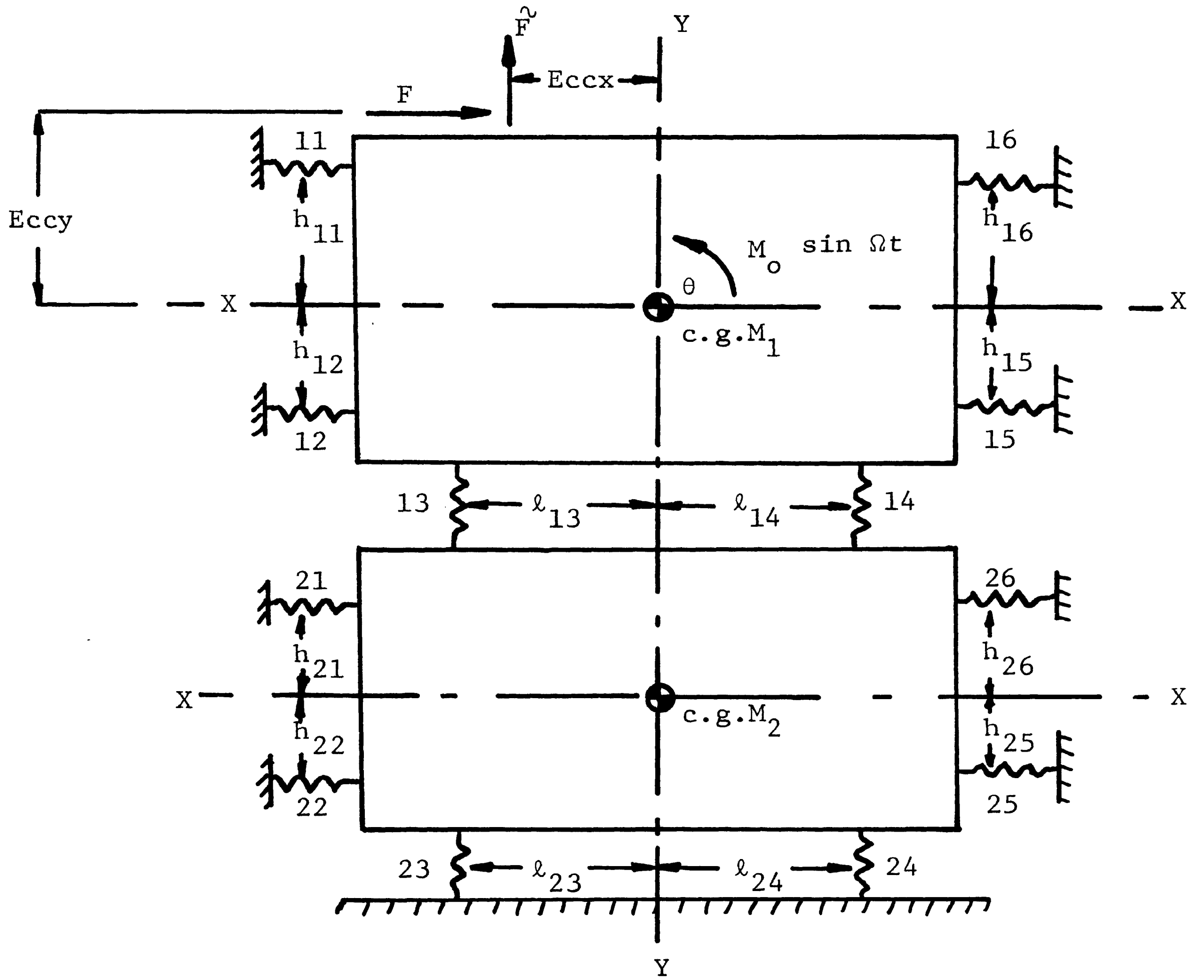


Fig. 2.6 The Two Mass System

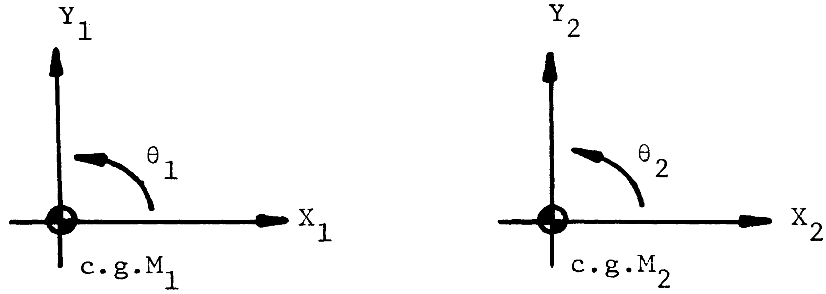


Fig. 2.7 Sign Conventions for the Two Mass System

gravity of each mass. Figure 2.8 shows the free body diagrams for masses M_1 and M_2. Applying Newton's second law, the differential equations of motion are:

$$\begin{aligned}
M_1\ddot{X}_1 = &- k_{11}(X_1-h_{11}\theta_1) - k_{12}(X_1+h_{12}\theta_1) - [k_{13}(X_1+h_{13}\theta_1)- \\
&k_{13}(X_2-h_2\theta_2)] - [k_{14}(X_1+h_{14}\theta_1)-k_{14}(X_2-h_2\theta_2)] \\
&- k_{15}(X_1+h_{15}\theta_1) - k_{16}(X_1-h_{16}\theta_1) + F\sin\Omega t.
\end{aligned}$$

$$\begin{aligned}
M_1\ddot{Y}_1 = &- \tilde{k}_{11}(Y_1-\ell_{11}\theta_1) - \tilde{k}_{12}(Y_1-\ell_{12}\theta_1) - [\tilde{k}_{13}(Y_1-\ell_{13}\theta_1)- \\
&\tilde{k}_{13}(Y_2-\ell_{13}\theta_2)] - [\tilde{k}_{14}(Y_1+\ell_{14}\theta_1)-\tilde{k}_{14}(Y_2+\ell_{14}\theta_2)] \\
&- \tilde{k}_{15}(Y_1+\ell_{15}\theta_1) - \tilde{k}_{16}(Y_1+\ell_{16}\theta_1) + \tilde{F}\sin\Omega t.
\end{aligned}$$

$$\begin{aligned}
I_1\ddot{\theta}_1 = &\ \tilde{k}_{11}\ell_{11}(Y_1-\ell_{11}\theta_1) + \tilde{k}_{12}\ell_{12}(Y_1-\ell_{12}\theta_1) + [\tilde{k}_{13}\ell_{13} \\
&(Y_1-\ell_{13}\theta_1)-\tilde{k}_{13}\ell_{13}(Y_2-\ell_{13}\theta_2)] - [\tilde{k}_{14}\ell_{14}(Y_1+\ell_{14}\theta_1)- \\
&\tilde{k}_{14}\ell_{14}(Y_2+\ell_{14}\theta_2)] - \tilde{k}_{15}\ell_{15}(Y_1+\ell_{15}\theta_1) - \tilde{k}_{16}\ell_{16} \\
&(Y_1+\ell_{16}\theta_1) + k_{11}h_{11}(X_1-h_{11}\theta_1) - k_{12}h_{12}(X_1+h_{12}\theta_1) \\
&- [k_{13}h_{13}(X_1+h_{13}\theta_1)-k_{13}h_{13}(X_2-h_2\theta_2)] - [k_{14}h_{14} \\
&(X_1+h_{14}\theta_1)-k_{14}h_{14}(X_2-h_2\theta_2)] - k_{15}h_{15}(X_1+h_{15}\theta_1) \\
&+ k_{16}h_{16}(X_1-h_{16}\theta_1) + \tilde{M}_o\sin\Omega t.
\end{aligned}$$

$$\begin{aligned}
M_2\ddot{X}_2 = &- k_{21}(X_2-h_{21}\theta_2) - k_{22}(X_2+h_{22}\theta_2) - k_{23}(X_2+h_{23}\theta_2) \\
&- k_{24}(X_2+h_{24}\theta_2) - k_{25}(X_2+h_{25}\theta_2) - k_{26}(X_2-h_{26}\theta_2) \\
&+ [k_{13}(X_1+h_{13}\theta_1)-k_{13}(X_2-h_2\theta_2)] + [k_{14}(X_1+h_{14}\theta_1)- \\
&k_{14}(X_2-h_2\theta_2)].
\end{aligned}$$

$$M_2\ddot{Y}_2 = - \tilde{k}_{21}(Y_2-\ell_{21}\theta_2) - \tilde{k}_{22}(Y_2-\ell_{22}\theta_2) - \tilde{k}_{23}(Y_2-\ell_{23}\theta_2)$$

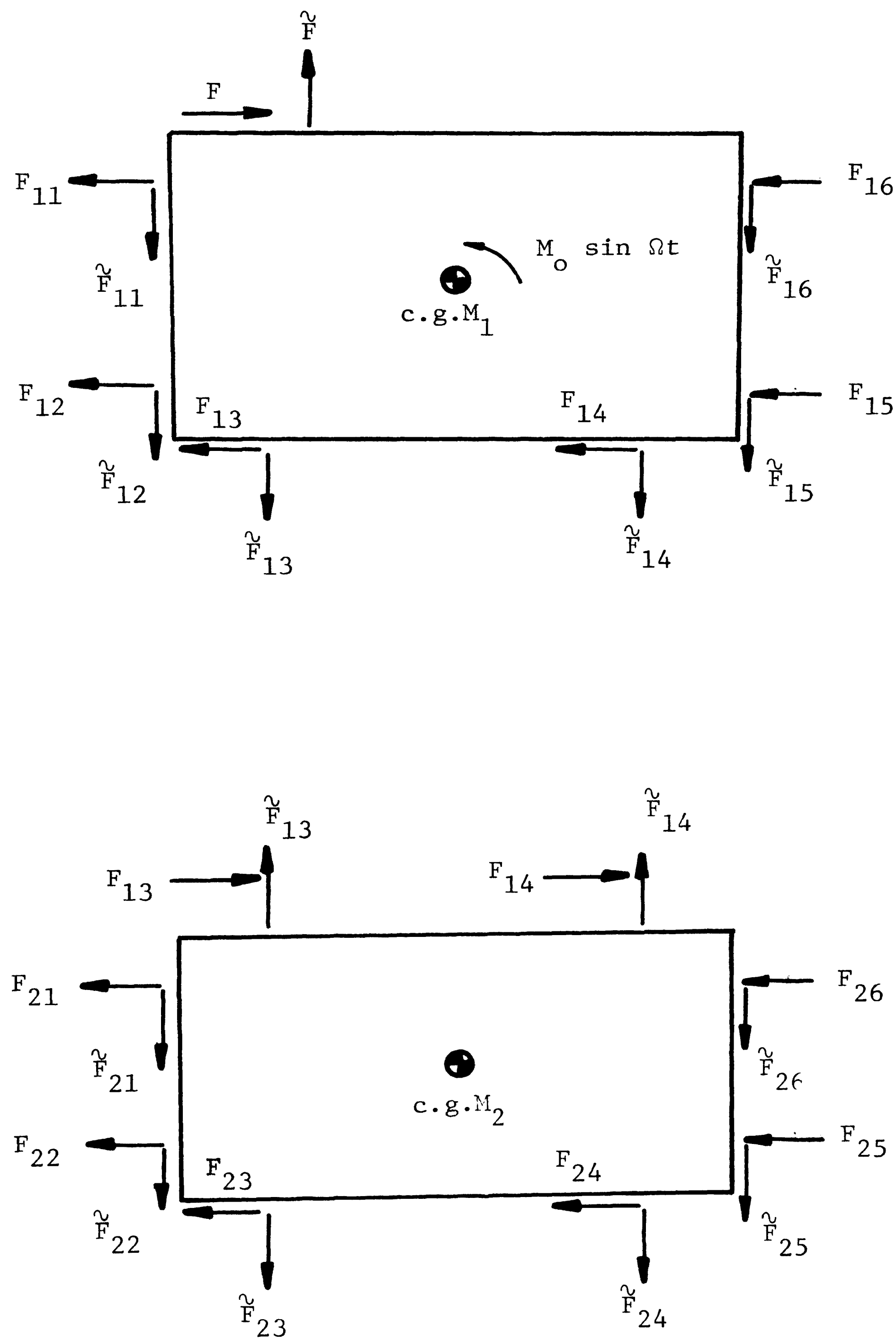


Fig. 2.8 Free Body Diagram for the Two Mass System

In Figure (2.8):

$$F_{11} = k_{11}(X_1-h_{11}\theta_1)$$

$$\tilde{F}_{11} = \tilde{k}_{11}(Y_1-\ell_{11}\theta_1)$$

$$F_{12} = k_{12}(X_1+h_{12}\theta_1)$$

$$\tilde{F}_{12} = \tilde{k}_{12}(Y_1-\ell_{12}\theta_1)$$

$$F_{13} = k_{13}(X_1+h_{13}\theta_1) - k_{13}(X_2-h_2\theta_2)$$

$$\tilde{F}_{13} = \tilde{k}_{13}(Y_1-\ell_{13}\theta_1) - \tilde{k}_{13}(Y_2-\ell_{13}\theta_2)$$

$$F_{14} = k_{14}(X_1+h_{14}\theta_1) - k_{14}(X_2-h_2\theta_2)$$

$$\tilde{F}_{14} = \tilde{k}_{14}(Y_1+\ell_{14}\theta_1) - \tilde{k}_{14}(Y_2+\ell_{14}\theta_2)$$

$$F_{15} = k_{15}(X_1+h_{15}\theta_1)$$

$$\tilde{F}_{15} = \tilde{k}_{15}(Y_1+\ell_{15}\theta_1)$$

$$F_{16} = k_{16}(X_1-h_{16}\theta_1)$$

$$\tilde{F}_{16} = \tilde{k}_{16}(Y_1+\ell_{16}\theta_1)$$

$$F_{21} = k_{21}(X_2-h_{21}\theta_2)$$

$$\tilde{F}_{21} = \tilde{k}_{21}(Y_2-\ell_{21}\theta_2)$$

$$F_{22} = k_{22}(X_2+h_{22}\theta_2)$$

$$\tilde{F}_{22} = \tilde{k}_{22}(Y_2-\ell_{22}\theta_2)$$

$$F_{23} = k_{23}(X_2+h_{23}\theta_2)$$

$$\tilde{F}_{23} = \tilde{k}_{23}(Y_2-\ell_{23}\theta_2)$$

$$F_{24} = k_{24}(X_2+h_{24}\theta_2)$$

$$\tilde{F}_{24} = \tilde{k}_{24}(Y_2+\ell_{24}\theta_2)$$

$$F_{25} = k_{25}(X_2+h_{25}\theta_2)$$

$$\tilde{F}_{25} = \tilde{k}_{25}(Y_2+\ell_{25}\theta_2)$$

$$F_{26} = k_{26}(X_2-h_{26}\theta_2)$$

$$\tilde{F}_{26} = \tilde{k}_{26}(Y_2+\ell_{26}\theta_2)$$

$$- \tilde{k}_{24}(Y_2 + \ell_{24}\theta_2) - \tilde{k}_{25}(Y_2+\ell_{25}\theta_2) - \tilde{k}_{26}(Y_2+\ell_{26}\theta_2)$$
$$+ [\tilde{k}_{13}(Y_1-\ell_{13}\theta_1)-\tilde{k}_{13}(Y_2-\ell_{13}\theta_2)] + [\tilde{k}_{14}(Y_1+\ell_{14}\theta_1)-$$
$$\tilde{k}_{14}(Y_2+\ell_{14}\theta_2)].$$

$$I_2\ddot{\theta}_2 = \tilde{k}_{21}\ell_{21}(Y_2-\ell_{21}\theta_2) + \tilde{k}_{22}\ell_{22}(Y_2-\ell_{22}\theta_2) + \tilde{k}_{23}\ell_{23}$$
$$(Y_2-\ell_{23}\theta_2) - \tilde{k}_{24}\ell_{24}(Y_2+\ell_{24}\theta_2) - \tilde{k}_{25}\ell_{25}(Y_2+\ell_{25}\theta_2)$$
$$- \tilde{k}_{26}\ell_{26}(Y_2+\ell_{26}\theta_2) - [\tilde{k}_{13}\ell_{13}(Y_1-\ell_{13}\theta_1)-\tilde{k}_{13}\ell_{13}$$
$$(Y_2-\ell_{13}\theta_2)] + [\tilde{k}_{14}\ell_{14}(Y_1+\ell_{14}\theta_1)-\tilde{k}_{14}\ell_{14}(Y_2+\ell_{14}\theta_2)]$$
$$+ k_{21}h_{21}(X_2-h_{21}\theta_2) - k_{22}h_{22}(X_2+h_{22}\theta_2) - k_{23}h_{23}$$
$$(X_2+h_{23}\theta_2) - k_{24}h_{24}(X_2+h_{24}\theta_2) - k_{25}h_{25}(X_2+h_{25}\theta_2)$$
$$+ k_{26}h_{26}(X_2-h_{26}\theta_2) - [k_{13}h_2(X_1+h_{13}\theta_1)-k_{13}h_2$$
$$(X_2-h_2\theta_2)] - [k_{14}h_2(X_1+h_{14}\theta_1)-k_{14}h_2(X_2-h_2\theta_2)].$$

Simplifying, the equations give:

$$M_1\ddot{X}_1 + (\sum_j k_{1j})X_1 - (\sum_{cs} k_{1cs})X_2 - (\sum_j k_{1j}h_{1j})\theta_1 + (\sum_{cs} k_{1cs}h_2)\theta_2 = F \sin \Omega t$$

$$M_1\ddot{Y}_1 + (\sum_j \tilde{k}_{1j})Y_1 - (\sum_{cs} \tilde{k}_{1cs})Y_2 + (\sum_j \tilde{k}_{1j}\ell_{1j})\theta_1 - (\sum_{cs} \tilde{k}_{1cs}\ell_{1cs})\theta_2 = \tilde{F} \sin \Omega t$$

$$I_1\ddot{\theta}_1 - (\sum_j k_{1j}h_{1j})X_1 + (\sum_j \tilde{k}_{1j}\ell_{1j})Y_1 + [(\sum_j \tilde{k}_{1j}\ell_{1j}^2)+(\sum_j k_{1j}h_{1j}^2)]\theta_1$$
$$- (\sum_{cs} k_{1cs}h_{1cs})X_2 - (\sum_{cs} \tilde{k}_{1cs}\ell_{1cs})Y_2 - [(\sum_{cs} \tilde{k}_{1cs}\ell_{1cs}^2)-(\sum_{cs} k_{1cs}h_{1cs}h_2)]\theta_2$$
$$= \tilde{M}_o \sin \Omega t.$$

Here, cs = number of springs connecting M_1 and M_2.

$$M_2\ddot{X}_2 - (\sum_{cs} k_{1cs})X_1 + (\sum_j k_{2j}+\sum_{cs} k_{1cs})X_2 - (\sum_{cs} k_{1cs}h_{1cs})\theta_1$$
$$- [(\sum_j k_{2j}h_{2j})+(\sum_{cs} k_{1cs}h_2)]\theta_2 = 0$$

$$M_2\ddot{Y}_2 - (\sum_{cs}\tilde{k}_{1cs})Y_1 + [(\sum_j\tilde{k}_{2j}+\sum_{cs}\tilde{k}_{1cs})]Y_2 - (\sum_{cs}\tilde{k}_{1cs}\ell_{1cs})\theta_1$$
$$+ [(\sum_j\tilde{k}_{2j}\ell_{2j}+\sum_{cs}\tilde{k}_{1cs}\ell_{1cs})]\theta_2 = 0$$

$$I_2\ddot{\theta}_2 + (\sum_{cs}k_{1cs}h_2)X_1 - (\sum_{cs}\tilde{k}_{1cs}\ell_{1cs})Y_1 - [(\sum_{cs}\tilde{k}_{1cs}\ell^2_{1cs})-(\sum_{cs}k_{1cs}h_{1cs}h_2)]\theta_1$$
$$- [(\sum_j k_{2j}h_{2j})+(\sum_{cs}k_{1cs}h_2)]X_2 + [(\sum_j\tilde{k}_{2j}\ell_{2j})+(\sum_{cs}\tilde{k}_{1cs}\ell_{1cs})]Y_2 + [(\sum_j\tilde{k}_{2j}\ell^2_{2j})$$
$$+(\sum_j k_{2j}h^2_{2j})+(\sum_{cs}\tilde{k}_{1cs}\ell^2_{1cs})+\sum_{cs}k_{1cs}h^2_2]\theta_2 = 0$$

Writing in general matrix form:

$$[\ M_2\ ]\{\ddot{\eta}\} + [K_2]\{\eta\} = \{F\}. \tag{2.2}$$

where:

$$[\ M_2\ ] = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 \end{bmatrix}$$

$$\{\ddot{\eta}\} = \begin{Bmatrix} \ddot{X}_1 \\ \ddot{Y}_1 \\ \ddot{\theta}_1 \\ \ddot{X}_2 \\ \ddot{Y}_2 \\ \ddot{\theta}_2 \end{Bmatrix} \quad \text{and,} \quad \{\eta\} = \begin{Bmatrix} X_1 \\ Y_1 \\ \theta_1 \\ X_2 \\ Y_2 \\ \theta_2 \end{Bmatrix}$$

The stiffness matrix $[K_2]$ is a symmetrical matrix of order six. It can be divided into four submatrices each of order three as shown below:

$$[K_2] = \left[\begin{array}{c|c} K_{2A} & K_{2B} \\ \hline K_{2C} & K_{2D} \end{array}\right]$$

The elements of submatrix $[K_{2A}]$ are identical to the elements of $[K_1]$. The remaining upper triangular elements of $[K_2]$ are:

$$K_{14} = -\sum_{cs} k_{1cs}$$

$$K_{15} = 0$$

$$K_{16} = \sum_{cs} k_{1cs} h_2$$

$$K_{24} = 0$$

$$K_{25} = -\sum_{cs} \tilde{k}_{1cs}$$

$$K_{26} = -\sum_{cs} \tilde{k}_{1cs} \ell_{1cs}$$

$$K_{34} = -\sum_{cs} k_{1cs} h_{1cs}$$

$$K_{35} = -\sum_{cs} \tilde{k}_{1cs} \ell_{1cs}$$

$$K_{36} = -\sum_{cs} \tilde{k}_{1cs} \ell_{1cs}^2 + \sum_{cs} k_{1cs} h_{1cs} h_2$$

$$K_{44} = \sum_{j} k_{2j} + \sum_{cs} k_{1cs}$$

$$K_{45} = 0$$

$$K_{46} = -\sum_{j} k_{2j} h_{2j} - \sum_{cs} k_{1cs} h_2$$

$$K_{55} = \sum_{j} \tilde{k}_{2j} + \sum_{cs} \tilde{k}_{1cs}$$

$$K_{56} = \sum_{j} \tilde{k}_{2j} \ell_{2j} + \sum_{cs} \tilde{k}_{1cs} \ell_{1cs}$$

$$K_{66} = \sum_{j} \tilde{k}_{2j} \ell_{2j}^2 + \sum_{j} k_{2j} h_{2j}^2 + \sum_{cs} \tilde{k}_{1cs} \ell_{1cs}^2 + \sum_{cs} k_{1cs} h_2^2 \quad .$$

And, the forcing function vector for this system is:

$$\{F\} = \begin{Bmatrix} F & \sin \Omega t \\ \tilde{F} & \sin \Omega t \\ \tilde{M}_o & \sin \Omega t \\ & 0 \\ & 0 \\ & 0 \end{Bmatrix}.$$

D. <u>Three Mass System and Governing Differential Equations</u>

Figure 2.9 shows a typical three mass system. The masses M_1 and M_2 are considered to have the same generalized displacements as the two mass system while mass M_3 has displacements X_3, Y_3 and θ_3. The displacements of M_2 are taken to be larger in magnitude than of M_3. Here again this is done for convenience in writing the equations of motion. The external forces and moments act only on M_1 as in previous cases.

The sign conventions used for displacements, distances, forces and moments are the same as used previously in the one and two mass systems.

Equations of motions now can be written easily after inspecting the equations of the two mass system. Therefore, the general matrix equation is:

$$[\,M_3\,]\,\{\ddot{\eta}\} + [K_3]\{\eta\} = \{F\}. \qquad (2.3)$$

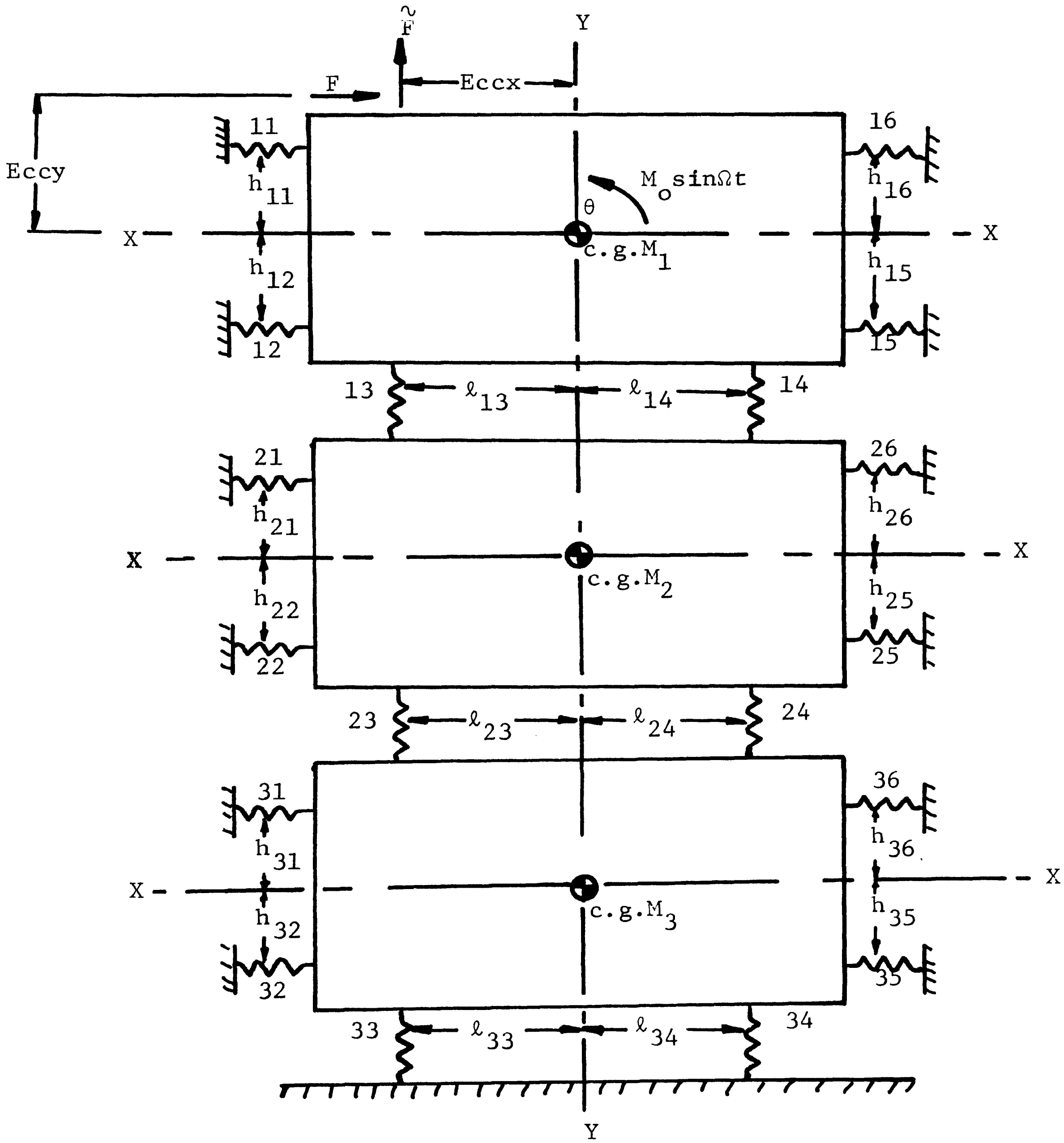


Fig. 2.9 Adjacently Connected Three Mass System

where:

$$[M_3] = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \end{bmatrix}$$

$$\{\ddot{n}\} = \begin{Bmatrix} \ddot{X}_1 \\ \ddot{Y}_1 \\ \ddot{\theta}_1 \\ \ddot{X}_2 \\ \ddot{Y}_2 \\ \ddot{\theta}_2 \\ \ddot{X}_3 \\ \ddot{Y}_3 \\ \ddot{\theta}_3 \end{Bmatrix} \quad \text{and,} \quad \{n\} = \begin{Bmatrix} X_1 \\ Y_1 \\ \theta_1 \\ X_2 \\ Y_2 \\ \theta_2 \\ X_3 \\ Y_3 \\ \theta_3 \end{Bmatrix} .$$

The stiffness matrix $[K_3]$ is a symmetrical matrix of order nine. It can be partitioned into nine submatrices each of order three as shown below:

$$[K_3] = \left[\begin{array}{c|c|c} K_{3A} & K_{3B} & K_{3C} \\ \hline K_{3D} & K_{3E} & K_{3F} \\ \hline K_{3G} & K_{3H} & K_{3I} \end{array}\right]$$

The submatrices K_{3A}, K_{3B}, K_{3D} and K_{3E} form a matrix of order six whose elements are identical to those of $[K_2]$. The remaining upper triangular elements of $[K_3]$ are:

$$[K_{3C}] = 0$$

$$K_{47} = -\sum_{cs} k_{2cs}$$

$$K_{48} = 0$$

$$K_{49} = \sum_{cs} k_{2cs} h_3$$

$$K_{57} = 0$$

$$K_{58} = -\sum_{cs} \tilde{k}_{2cs}$$

$$K_{59} = -\sum_{cs} \tilde{k}_{2cs} \ell_{2cs}$$

$$K_{67} = -\sum_{cs} k_{2cs} h_2$$

$$K_{68} = -\sum_{cs} \tilde{k}_{2cs} \ell_{2cs}$$

$$K_{69} = -\sum_{cs} \tilde{k}_{2cs} \ell_{2cs}^2 + \sum_{cs} k_{2cs} h_2 h_3$$

$$K_{77} = \sum_{j} k_{3j} + \sum_{cs} k_{2cs}$$

$$K_{78} = 0$$

$$K_{79} = -\sum_{j} k_{3j} h_{3j} - \sum_{cs} k_{2cs} h_3$$

$$K_{88} = \sum_{j} \tilde{k}_{3j} + \sum_{cs} \tilde{k}_{2cs}$$

$$K_{89} = \sum_{j} \tilde{k}_{3j} \ell_{3j} + \sum_{cs} \tilde{k}_{2cs} \ell_{2cs}$$

$$K_{99} = \sum_{j} \tilde{k}_{3j} \ell_{3j}^2 + \sum_{j} k_{3j} h_{3j}^2 + \sum_{cs} \tilde{k}_{2cs} \ell_{2cs}^2 + \sum_{cs} k_{2cs} h_{2cs}^2$$

Here, cs is the number of isolators between M_2 and M_3.

And, the forcing function vector is:

$$\{F\} = \begin{Bmatrix} F \ \sin \Omega t \\ \tilde{F} \ \sin \Omega t \\ \tilde{M}_o \ \sin \Omega t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} .$$

Some cases in the two mass system have isolators connected directly to the foundation from the top mass while some isolators connect the two adjacent masses in series as shown in Figure 2.10. Similarly, in the three mass system some cases have isolators connected to the foundation directly from the top or from the middle mass while other isolators connect the three adjacent masses in series.

For these cases the matrix equation of motion remains the same as established earlier, but changes will occur in stiffness matrix elements. The changes in $[K_2]$ or $[K_3]$ will be in submatrices whose elements are associated with the mass from which the isolators go to the foundation. The summation in each element increases due to the fact that, now the summation has to be done for more isolators. For the case in Figure 2.10 the submatrix $[K_{2A}]$ of stiffness matrix $[K_2]$ will change. Here the summation for each element will be for

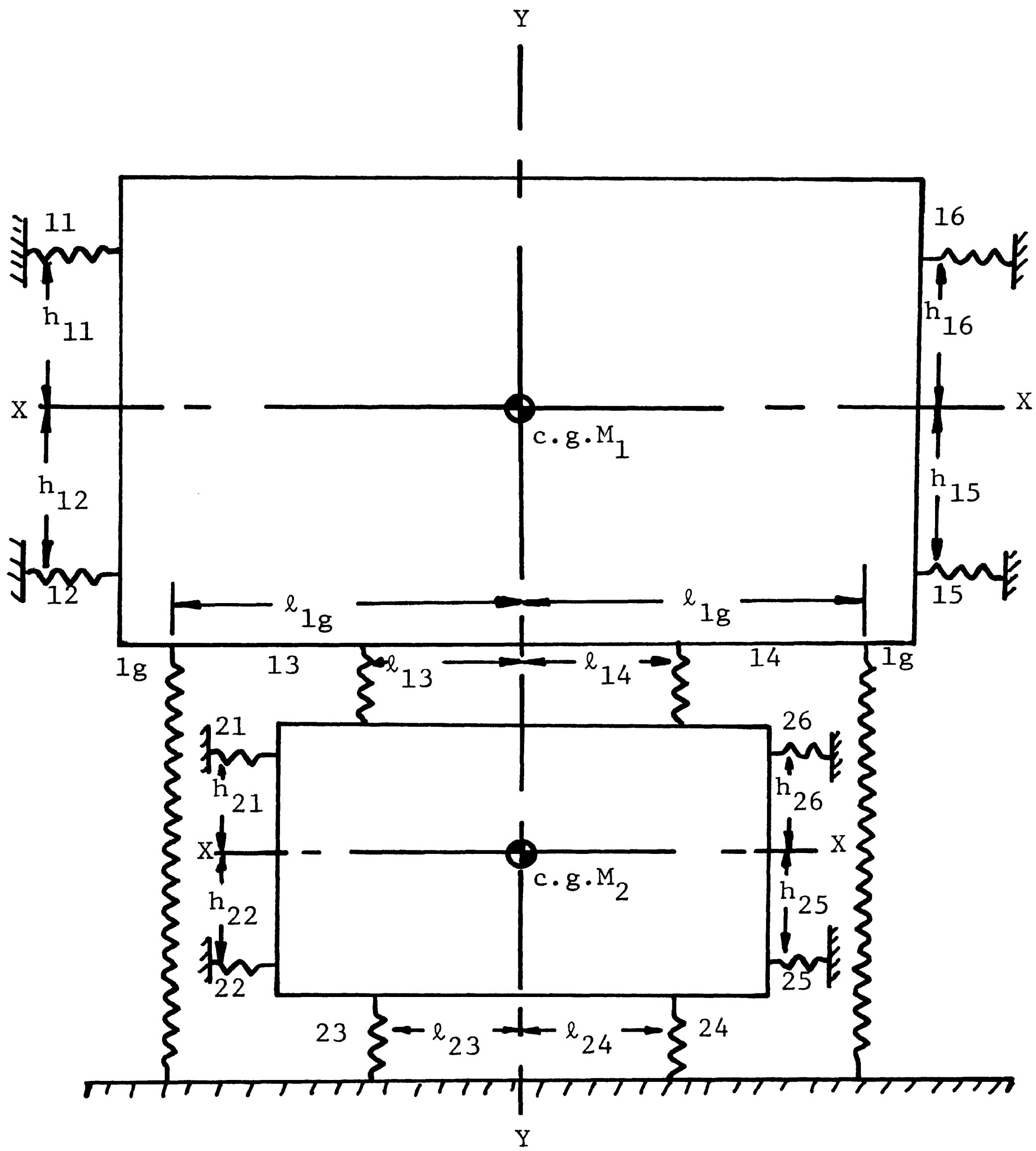


Fig. 2.10 Two Mass System with Springs to the Foundation from Top Mass

the isolators going to the ground as well as for the isolators connecting M_2. All the elements in other submatrices remain the same.

In the three mass system where some isolators are connected directly between M_1 and M_3 and the other isolators connect the three adjacent masses in series as shown in Figure 2.11, the stiffness matrix $[K_3]$ shows changes in the elements as shown below:

$$K_{17} = -\sum_{cs} k_{3cs}$$

$$K_{19} = \sum_{cs} k_{3cs} h_3$$

$$K_{28} = -\sum_{cs} \tilde{k}_{3cs}$$

$$K_{29} = -\sum_{cs} \tilde{k}_{3cs} \ell_{3cs}$$

$$K_{37} = -\sum_{cs} k_{3cs} h_1$$

$$K_{38} = \sum_{cs} - \tilde{k}_{3cs} \ell_{3cs}$$

$$K_{39} = -\sum_{cs} \tilde{k}_{3cs} \ell_{3cs}^2 + \sum_{cs} k_{3cs} h_1 h_3$$

$$K_{77} = K_{77} + \sum_{cs} K_{3cs}$$

$$K_{79} = K_{79} - \sum_{cs} K_{3cs} h_3$$

$$K_{88} = K_{88} + \sum_{cs} \tilde{k}_{3cs}$$

$$K_{89} = K_{89} + \sum_{cs} \tilde{k}_{3cs} \ell_{3cs}$$

$$K_{99} = K_{99} + \sum_{cs} \tilde{k}_{3cs} \ell_{3cs}^2 + \sum_{cs} K_{3cs} h_{3cs}^2 .$$

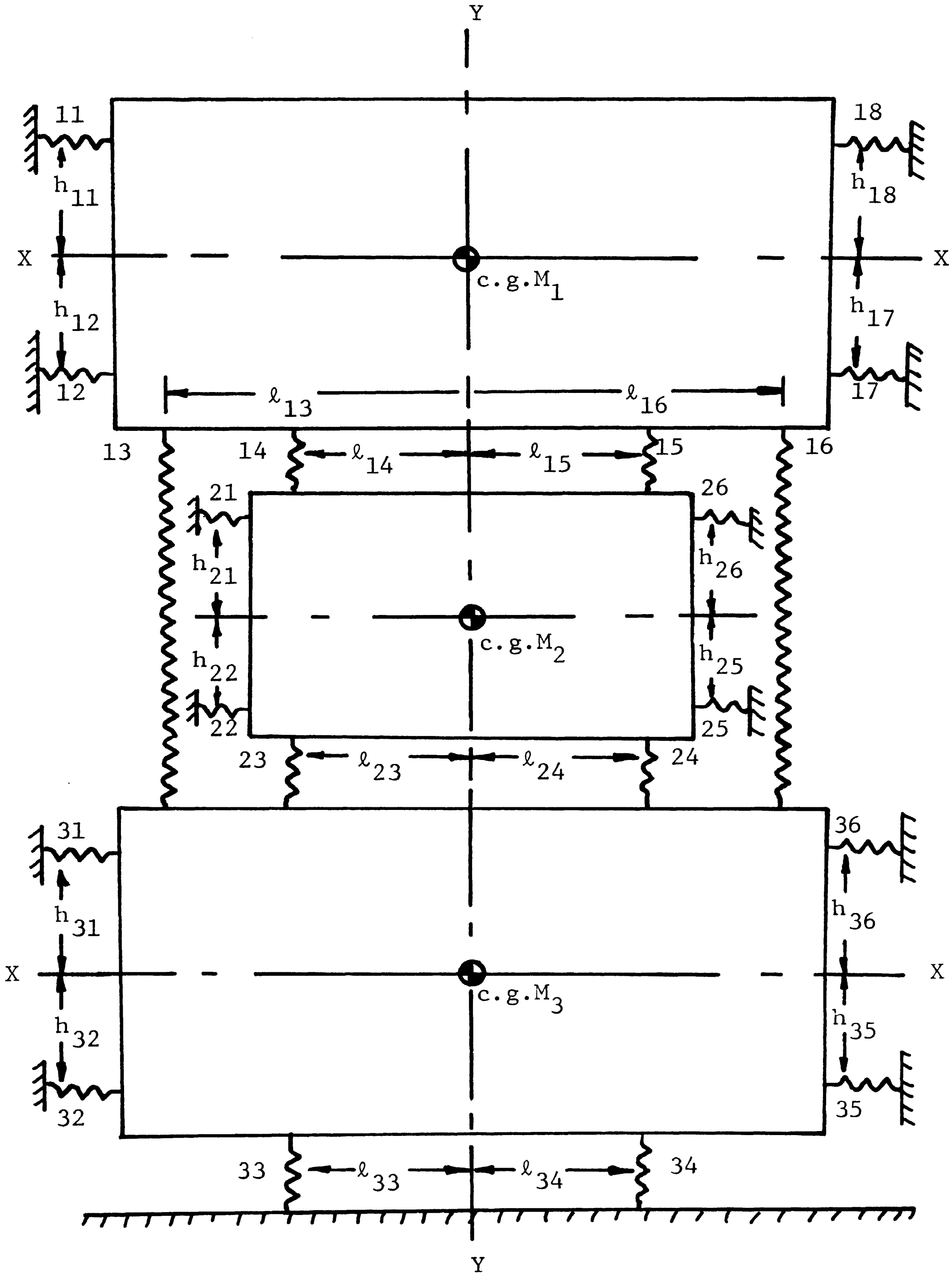


Fig. 2.11 Three Mass System with Springs between M_1 and M_3

Here, cs is the number of isolators connecting M_1 and M_3.

It is to be noted that equations (2.1), (2.2) and (2.3) have the same matrix form. The difference in the matrix differential equation for each system is only that the matrix represents a different number of differential equations. Solutions to these equations are established in Chapter III in matrix form and will be applied to evaluate the effectiveness of the various isolation systems. The objective is to determine what advantages might be gained in using two or three mass systems as opposed to the conventional single inertial mass isolation system.

CHAPTER III

SOLUTIONS TO THE EQUATIONS OF MOTION

A. Homogeneous Solution

The governing differential equations for the three cascaded systems to be examined herein were established in Chapter II. Their general matrix form is the same in each case. The solutions established in this chapter apply to each of the isolation systems and to any other system whose governing differential equations are of the matrix form displayed in equations (2.1), (2.2) and (2.3).

The eigenvectors and eigenvalues are obtained from the homogeneous solution. They are used to establish the dynamic response solution by the standard superposition of normal modes transformation [6]. By this method the differential equations are entirely uncoupled for any mode. Solutions to these uncoupled equations can be used to construct the overall time solution for the displacements of the system.

For the homogeneous solution, the differential equation of motion in matrix notation is given by:

$$[\,M\,]\,\{\ddot{\eta}\} + [K]\{\eta\} = \{0\}. \tag{3.1}$$

To retain symmetry in the final matrix form for the eigenvalue problem, the following transformation is used:

$$\{\eta\} = [\,M\,]^{-1/2}\{\tilde{\eta}\}.$$

Substituting this into eq. (3.1) and premultiplying by $[\,M\,]^{-1/2}$ gives:

$$[\,M\,]^{-1/2}[\,M\,][\,M\,]^{-1/2}\{\ddot{\tilde{\eta}}\} + [\,M\,]^{-1/2}[K][\,M\,]^{-1/2}\{\tilde{\eta}\} = \{0\}. \tag{3.2}$$

But, $[M]^{-1/2}[M][M]^{-1/2} = [I]$ the identity matrix, and hence eq. (3.2) becomes:

$$\{\ddot{\tilde{\eta}}\} + [\overline{K}]\{\tilde{\eta}\} = \{0\}$$

where: $$[\overline{K}] = [M]^{-1/2}[K][M]^{-1/2}$$

If [B] is a non-singular matrix and [R] a symmetric matrix, then:

$$[B]^T[R][B] = [\tilde{R}]$$

where: $[\tilde{R}]$ is a symmetric matrix.

Hence from above it can be concluded that matrix $[\overline{K}]$ is also a symmetrical matrix. Now eigenvectors and eigenvalues of the matrix $[\overline{K}]$ can be found. A standard eigenvalue subroutine from the IBM-360-50 computer library was employed to find the eigenvectors and eigenvalues.

The eigenvalues of a system are invariant with respect to the coordinates used to describe its motion [6]. Hence the eigenvalues of $[\overline{K}]$ are the same as those of [K]. The modal matrix formed by writing columnwise the eigenvector of $[\overline{K}]$, is premultiplied by $[M]^{-1/2}$ to obtain the eigenvectors in the original generalized coordinates:

$$[A] = [M]^{-1/2}[\tilde{A}],$$

where: $[\tilde{A}]$ is the modal matrix of $[\overline{K}]$ and [A] is the modal matrix of equation (3.1).

B. Forced Excitation Solution

The eigenvalues (ω_i) and eigenvector modal matrix [A] obtained from the homogeneous solution are used to form the forced excitation solution. Once the modal matrix is obtained, it is convenient to normalize it and use it in the standard superposition of normal modes transformation. The eigenvector $\{A\}_i$ can be normalized as shown below.

If a_i is a normalization constant, the normalized vector $\{\phi\}_i$ which is generally used in this approach, becomes:

$$\{\phi\}_i = \{A\}_i a_i \text{, and} \tag{3.3}$$

$$\{\phi\}_i^T [\ M\ ] \{\phi\}_i = \bar{m} \tag{3.4}$$

where: $\bar{m}$ = a constant whose value is selected for the convenience of the problem on hand and has the dimension of mass.

Substituting eq. (3.3) into eq. (3.4) gives:

$$a_i \{A\}_i^T [\ M\ ] \{A\}_i a_i = \bar{m}$$

$$\therefore \quad a_i^2 = \frac{\bar{m}}{\{A\}_i^T [\ M\ ] \{A\}_i}$$

a_i can be found from the expression above, and, thus the modal matrix, $[\phi]$, can be obtained from eq. (3.3), or:

$$[\phi] = [A] \lceil a_n \rfloor$$

where: $\lceil a_n \rfloor$ has a_i as diagonal elements

The differential equation of motion in matrix notation for the cascaded systems subjected to excitation forces is given by:

$$[\ M\ ]\{\ddot{\eta}\} + [K]\{\eta\} = \{F\}. \tag{3.5}$$

In the superposition of normal modes approach, the displacement vector is expressed as a linear combination of the eigenvectors which is expressed by:

$$\{\eta(t)\} = [\phi]\{\xi(t)\}. \tag{3.6}$$

where: $\{\eta(t)\}$, the displacement vector is a time dependent function.

Substituting eq. (3.6) into eq. (3.5) and premultiplying by $[\phi]^T$ gives:

$$[\phi]^T[\ M\ ][\phi]\{\ddot{\xi}\} + [\phi]^T[K][\phi]\{\xi\} = [\phi]^T\{F\}. \tag{3.7}$$

But from eigenvalue theory [3] we know that the eigenvectors in the system coordinates have orthogonality conditions through [M] and [K] of the form:

$$[\phi]^T[\ M\ ][\phi] = \bar{m}[\ulcorner I \lrcorner], \text{ and}$$
$$[\phi]^T[K][\phi] = \bar{m}[\ulcorner \omega_i^2 \lrcorner]$$

where: ω_i = natural frequency of the ith mode.

Using these relationships with eq. (3.7) gives:

$$\{\ddot{\xi}\} + [\ulcorner \omega_i^2 \lrcorner]\{\xi\} = \frac{[\phi]^T}{\bar{m}}\{F\}. \tag{3.8}$$

Equation (3.8) represents n uncoupled differential equations where n is the number of degrees of freedom. For the dynamic response of the inertial mass system only sinusoidal forces are considered. This would be typical of vibration exciters or many types of unbalanced machines. The sinusoidal forces in both the X and Y directions can be taken together or independently; in either case they are oscillating at the same frequency Ω. The solution to eq. (3.8) with sinusoidal forces [3] is:

$$\{\xi(t)\} = \frac{1}{\bar{m}}\left\{[\ulcorner \omega^2 \lrcorner] - \Omega^2[\ulcorner I \lrcorner]\right\}^{-1}[\phi]^T\{F\}. \tag{3.9}$$

Having the solution for $\{\xi(t)\}$, the time solution for the system displacements is:

$$\{\eta(t)\} = \frac{1}{m}[\phi]\left\{[\ulcorner\omega^2\lrcorner] - \Omega^2[\ulcorner I\lrcorner]\right\}^{-1}[\phi]^T\{F\}. \tag{3.10}$$

Equation (3.10) gives time dependent displacements or solution sought. The maximum displacements of the c.g. of each mass can be obtained by evaluating equation (3.10) over several cycles of each given Ω and searching for the maximum values.

C. Maximum Forces and Moments Transmitted to the Foundation

The force transmitted through each isolator is calculated as a function of displacement. The displacement of an isolator in the X and Y directions can be calculated once the displacements of the c.g. of each mass are known. This is shown in Figure 2.3 for the one mass system. Similarly, displacements of isolators connected to the foundation in the two and the three mass systems can be found. In general, the displacements can be written as:

$$X_{ij}(t) = X_i(t) - h_{ij}\theta_i(t), \text{ and}$$

$$Y_{ij}(t) = Y_i(t) + \ell_{ij}\theta_i(t)$$

where: $i = 1,2,3$ - designating the mass

$j = 1,2,3,\ldots,n$ - designating the isolator.

Therefore the forces at the isolator are:

$$F_{ij}(t) = k_{ij} \quad X_{ij}(t), \text{ and}$$

$$\tilde{F}_{ij}(t) = \tilde{k}_{ij} \quad Y_{ij}(t).$$

The total forces and moments transmitted to the foundation are:

$$F_T(t) = \sum_j F_{ij}(t)$$

$$\tilde{F}_T(t) = \sum_j \tilde{F}_{ij}(t), \text{ and}$$

$$M_T(t) = \sum_j (\tilde{F}_{ij}(t) \quad \ell_{ij}). \tag{3.11}$$

$F_T(t)$, $\tilde{F}_T(t)$ and $M_T(t)$ are searched over the time period to determine the maximum values for each forcing frequency Ω. Note that the moment is about the Z axis and is referred to the projection of the Z axis on the plane of the foundation. Equation (3.11) is the total moment referred to the c.g. projection in the foundation plane.

The force transmissibility is defined as the ratio of the total force transmitted by the isolators to the foundation, to the input force applied to the mass. It can be written as:

$$T_{XX} = \frac{F_T}{F},$$

$$T_{YY} = \frac{\tilde{F}_T}{\tilde{F}}, \text{ and}$$

$$T_{XY} = \frac{F_T}{\tilde{F}}.$$

The moment transmissibility can be written in a similar form as:

$$T_M = \frac{M_T}{M_o} .$$

D. Verification of the Equations of Motion

The equations of motion were verified by taking simple limiting cases, i.e., by choosing some springs to have zero or infinite stiffness. By choosing and placing the isolators properly, the degrees of freedom for any of the systems can be reduced and, hence, the number of equations are likewise reduced.

A computer program was written to evaluate the principal mode frequency roots and mode shapes, the forced excitation solutions, and maximum transmissibilities. This program was verified by using limiting cases as mentioned previously. The solutions obtained from the computer program were checked in limiting cases and at isolated points in time by hand calculations.

Limiting cases for the two and three mass systems were considered to check the validity of the computer program written for the solutions of the cascaded systems. Also, the form of the equations of motion was verified from this approach. As an illustration, a limiting case of the system shown in Figure 2.9 is considered. The isolators connecting masses M_1 and M_2, and M_2 and M_3 are made several orders of magnitude stiffer than the isolators connecting M_3 and the foundation. Also the horizontal springs connecting M_1, M_2 and M_3 to the foundation are taken to be zero. The isolators at the bottom are symmetrical about the Y axis and an external force is considered as applied through the c.g. of M_1.

As the isolators are kept symmetrical and the force is applied through the c.g. of M_1, the forced motion of the masses will be limited to the vertical direction. Hence, the system represents three masses connected by springs and having only vertical motion. The equations of motion of this simpler system were obtained and these equations were compared to the equations of motion (in the limiting case of symmetry) of the three mass system established earlier.

As the upper springs connecting M_1 and M_2, and M_2 and M_3 become infinitely rigid in the limiting case, the system in Figure 2.9 becomes an equivalent one mass system. The natural frequency of this system was calculated. The vertical motions of M_1 and M_2 will be approximately the same and hence, motion of all three masses can be considered the same. The motion of each mass can, thus, be calculated by hand and the results can be compared with the computer program results which do treat the general matrix problem of eq. (2.3) regardless of the size of the parameters.

By considering such limiting cases, the equations of motion were verified and the computer program was validated as the results for the fundamental frequency root and the mass displacements were very close.

CHAPTER IV

COMPARISON OF CASCADE SYSTEMS

A. Basis of Comparison

The main objective of this study has been to investigate various cascaded systems. These cases have been formulated by varying the combination of masses and isolators. The results obtained are compared to each other and then to the conventional one mass system.

Each system has been analyzed for its natural frequencies and mode shapes. The displacements of the c.g. of the top mass, on which the equipment for isolation is assumed to be attached, have also been analyzed. Finally, forces and moments transmitted by each system to the foundation have been determined and compared. The results from the analysis have been examined and compared to ascertain the possibility for improved vibration isolation.

For each system, the total weight has been kept constant. A total weight of four thousand pounds was chosen to be representative of actual isolation problems which might be encountered in industry.

The springs commonly used in these types of systems are air springs. They are usually made of rubber material. The most common types of air springs are bellows, roller sleeve and roller diaphragm types. In practice, the rubber bellows type springs have volumes and equivalent cross-sectional areas such that a system in which they are used has a natural frequency of approximately one cycle per second.

In this study, spring constants used are so selected as to have the natural frequencies of each mass on isolators to be one cps in all the systems. As an example, in Figure 2.11, M_1 with isolators 11, 12, 13, 14, 15, 16, 17 and 18, M_2 with isolators 21, 22, 23, 24, 25 and 26 and M_3 with isolators 31, 32, 33, 34, 35 and 36 are each selected to have a natural frequency of one cps. Stiffness coefficients for each isolator in the X and Y directions have been considered to be the same.

Exciting forces have been considered sinusoidal. The maximum amplitude of the exciting force has been fixed at one thousand pounds for the X and Y directions, respectively. This is well within the limits of practical cases. In practice the ratio of the exciting force to the total weight of the system is generally considered between 0.1 and 0.5. The distances where the applied forces on the systems are applied have been discussed in Amplitude Comparison under Forced Excitation.

Eighteen different cases of cascaded systems (see Appendix B) have been investigated. Two cases of the one mass system have been considered for reference as a more conventional system. In one case, the external force in the X direction is considered while in the other case the external force is in the Y direction.

Seven cases of the two mass system have been considered. In two cases, apart from the two masses being connected in series by isolators, some isolators from M_1 have been connected directly to the foundation (see Figure 2.10). The remaining five cases are typical two mass systems, adjacently connected mass and isolator combinations.

Nine cases have been considered in the three mass system. Four cases are typical adjacently connected three mass systems with variation in mass and stiffness distribution. In two cases (see Figure 2.11) some isolators connect M_1 and M_3 while the other isolators connect the three masses to the foundation in series. The remaining three cases have isolators going to the foundation from the top mass, the middle mass or from both the top two masses bypassing the bottom mass (see Figure A.1). Specific details of these cases are found in Appendix A, which gives the parametric values used.

B. Comparison of Natural Frequencies

It is important to identify the natural frequencies of the isolation systems to avoid any resonant conditions and to form the forced excitation solution. System displacements are inversely proportional to the difference between the natural frequencies squared and the forcing frequency squared; that is from eq. (3.9):

$$n_i \alpha \sum_i \frac{1}{(\omega_i^2 - \Omega^2)} \qquad (4.1)$$

Hence, it can be seen from eq. (4.1) that, if the natural frequencies of a system are as low as possible compared to the excitation frequency the motion of the mass in the system is reduced.

Any linear system with n degrees of freedom will have n natural frequencies which characterize the behavior of that system. Therefore, the one mass system has three natural frequencies, the two mass system has six, and the three mass system has nine. These natural frequencies are obtained from the eigenvalues of the system.

For comparison purposes, the bandwidth of the natural frequency roots for each case is formed. The bandwidth is the difference between the largest and smallest natural frequencies of a given system. It can be seen from eq. (4.1) that, the larger the bandwidth of a system, the smaller will be the amplitudes of the masses assuming that Ω lies above the largest ω_i.

As a result of this examination of the form of the solution, frequency bandwidth has been compared for all systems in hopes of establishing some simple criteria for isolation system comparison.

Figure 4.1 shows the natural frequency bandwidth for each case. Cases 1 and 2 (one mass system) have the smallest bandwidths. In the two mass system, cases 7 and 8 have the largest and the smallest bandwidths, respectively. For the three mass system cases 10 and 11 have the largest and smallest bandwidths, respectively.

The lowest and highest natural frequencies for case 1 range from 0.62 to 1.4, for case 8 from 0.49 to 1.44, for case 11 from 0.4 to 1.66, for case 7 from 0.28 to 2.27 and for case 10 from 0.18 to 2.51. Cases 1, 8 and 11 have bandwidths smaller than those of cases 7 and 10. It can also be noted from figure 4.1 that the difference in system configurations causes greater variation in the highest natural frequency among the cases than in the lowest natural frequency.

C. Comparison of Mode Shapes

Each principal mode is described by a natural frequency and a corresponding set of amplitudes which prescribe the relative motion of the mass or masses. These amplitudes can be normalized to any convenient form and are normalized here such that the sum of the amplitudes squared is always unity. These normalized mode shapes are the eigenvectors obtained from the homogeneous solution.

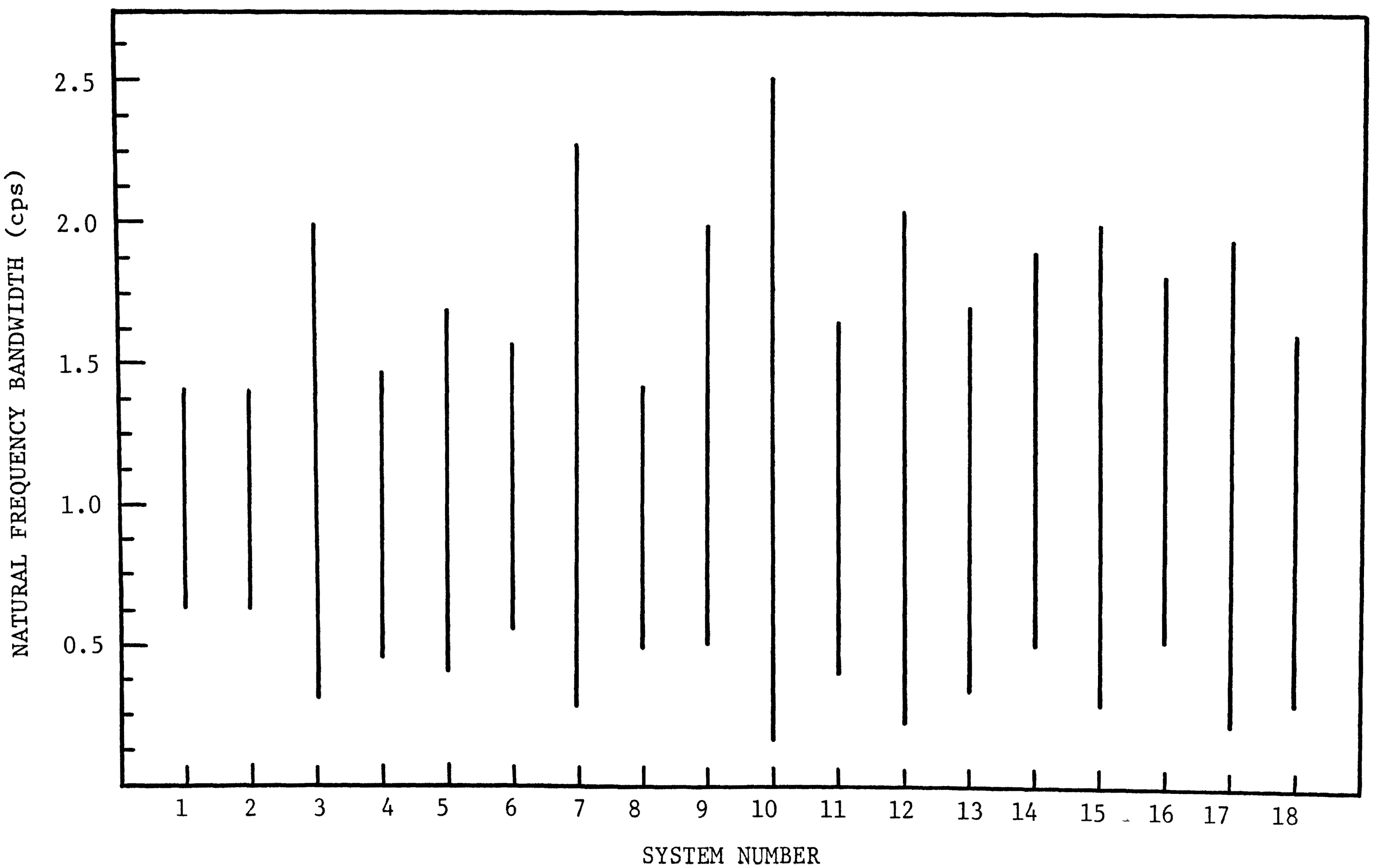


Fig. 4.1 Frequency Root Bandwidth of Cascaded Systems

Figures 4.2, 4.3 and 4.4 show six mode shapes for cases 4,3 and 6, respectively, in the two mass system. Figures 4.5 and 4.6 show nine mode shapes for each of the cases 11 and 14, respectively, in the three mass system. The sign conventions used for the mode shapes are the same as those used for the c.g. displacements as shown in Figure 2.4.

In the figures showing mode shapes, the short solid horizontal lines and the dark circles represent the masses at rest on the center line of the static equilibrium position. The dotted lines represent the displaced masses, i.e., displacement and slope, in the coupled modes. The unshaded circles show the displaced mass positions in the uncoupled modes which have Y displacements only, i.e., X and θ displacements are negligibly small in these modes. Because the systems have been considered to be symmetrical about the Y axis, there always occur one or more uncoupled vertical modes.

Cases 3 to 9 (see Appendices A and B) in the two mass cascaded system form three distinct groups which display mode shapes which are different in form from each other. The first group consists of cases 4, 5 and 8, the second group consists of cases 3 and 7 and the third group consists of cases 6 and 9. Figures 4.2, 4.3 and 4.4 give the mode shapes for the first, second and third group, respectively. In the first and second groups, the second and the fifth principal modes have only Y amplitudes. The other modes have the coupled X and θ amplitudes.

The first and the second groups also show dissimilarity in the fourth and the sixth modes while other modes display the similar

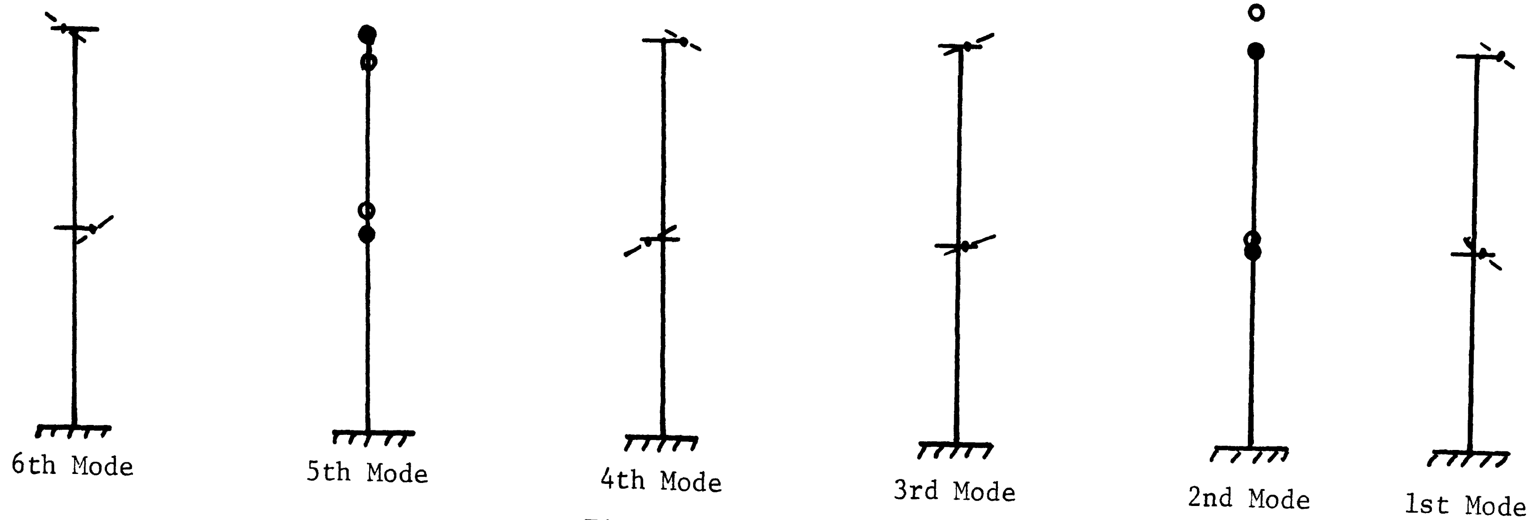


Fig. 4.2 Mode Shapes for Case 4

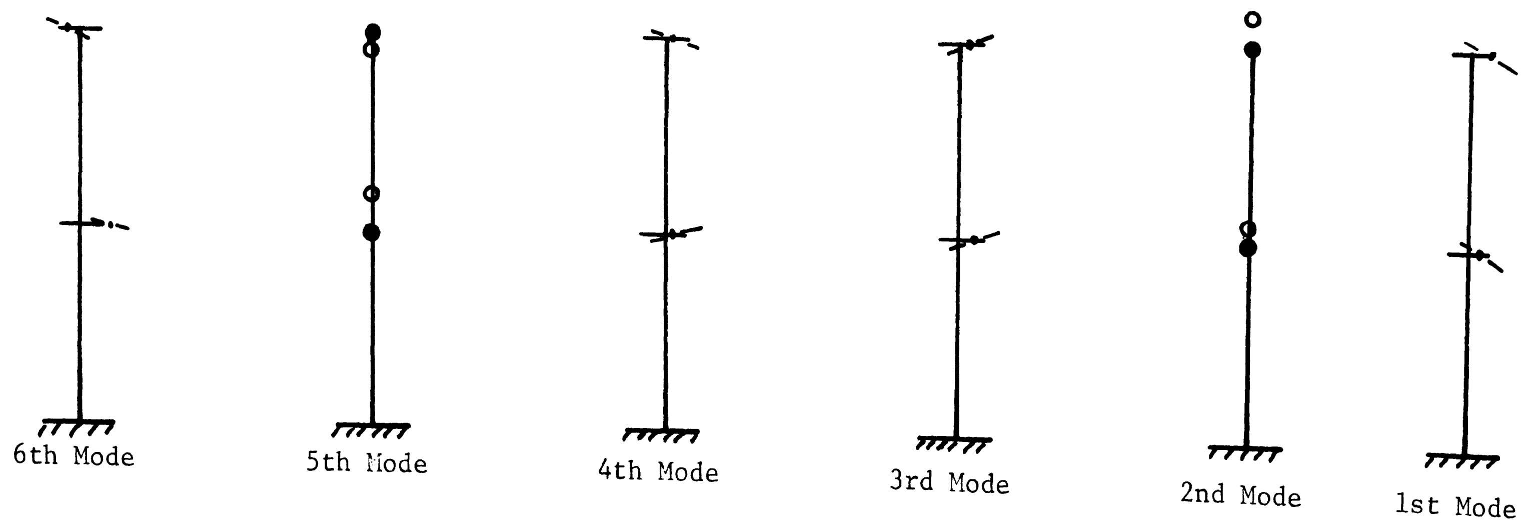
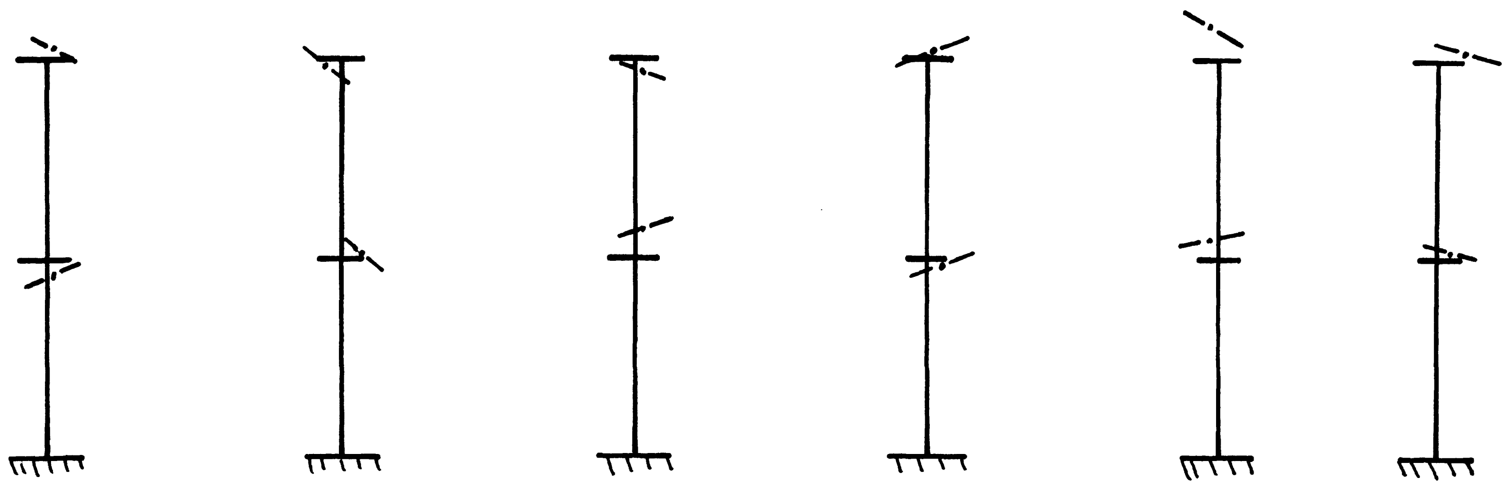


Fig. 4.3 Mode Shapes for Case 3



6th Mode 5th Mode 4th Mode 3rd Mode 2nd Mode 1st Mode

Fig. 4.4 Mode Shapes for Case 6

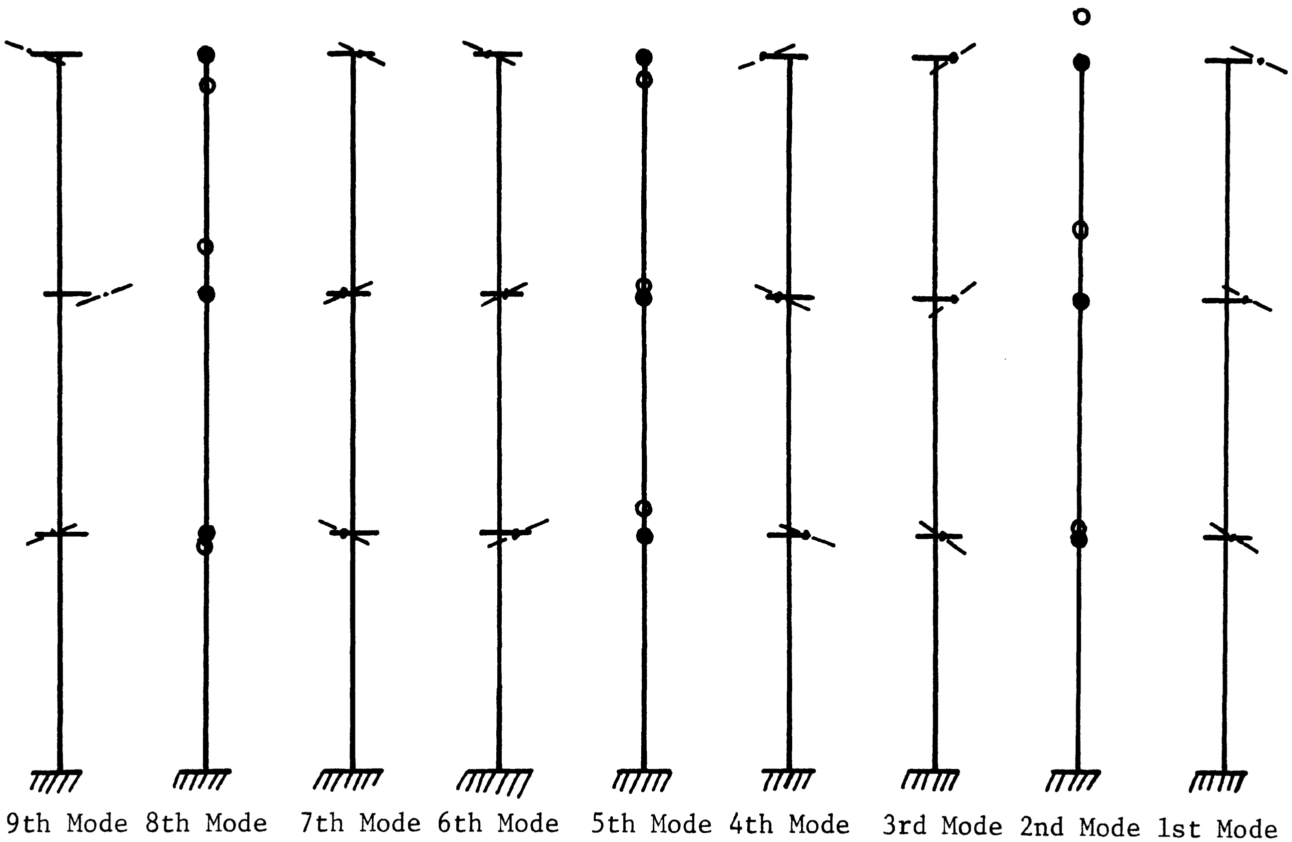


Fig. 4.5 Mode Shapes for Case 11

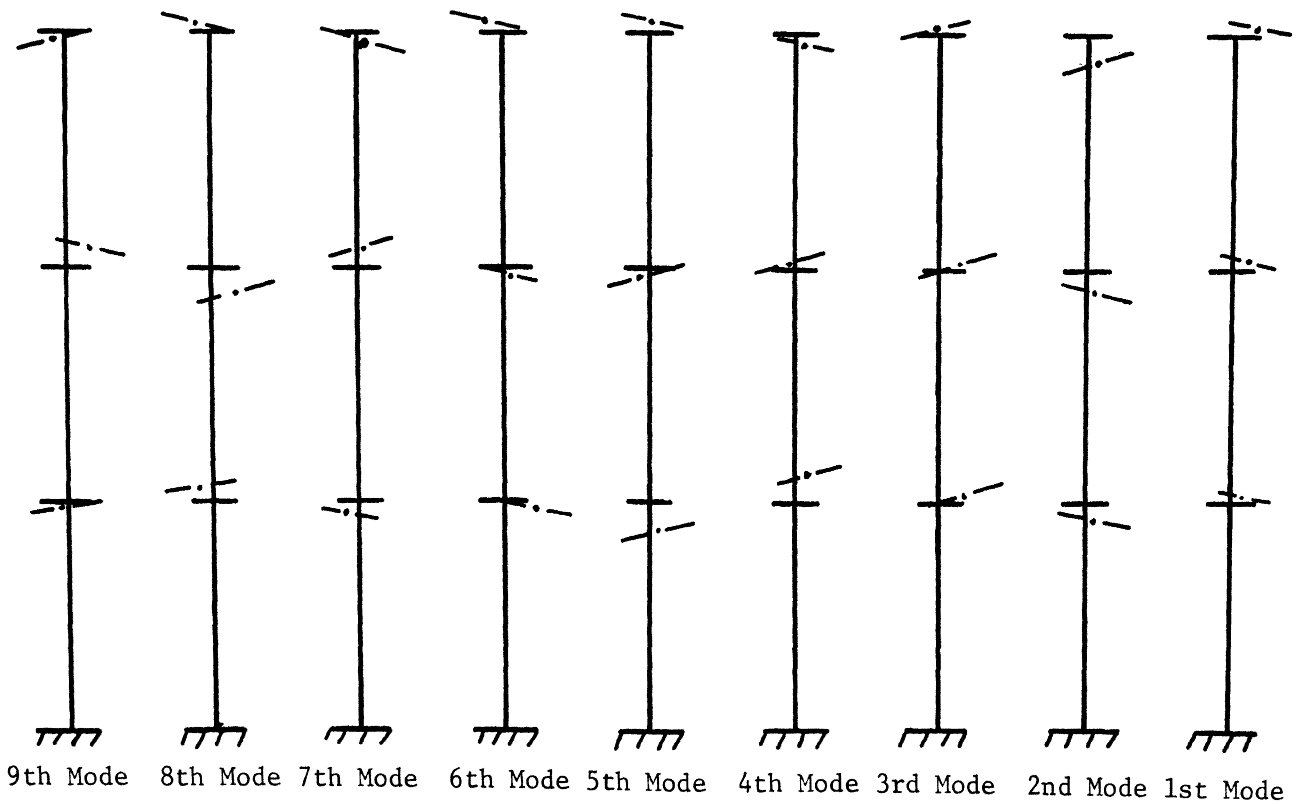


Fig. 4.6 Mode Shapes for Case 14

displacement patterns. For the first group in the fourth mode, the top and bottom masses move in opposite directions in both the X and θ amplitudes. For the second group in the same mode, both the masses move in the same directions in the X amplitudes while they move in opposite directions in the θ amplitudes. In the sixth mode for the first group, the two masses move in opposite directions in the X and θ amplitudes. For the second group in the same mode the two masses move in opposite directions in the X amplitudes while they move in the same direction in the θ amplitudes. The third group has no uncoupled Y mode, as occurs in groups one and two.

In the three mass systems represented by cases 9 to 18 (see Appendices A and B), three distinct groups can be found, each showing different mode shape patterns. In the first group cases 10 and 12 are included while cases 11 and 13 form the second group and the remaining cases form the third group. Figures 4.5 and 4.6 represent mode shapes for the second and the third group, respectively.

In the first two groups, the second, fifth and eighth modes have only Y displacements while the remaining modes have coupled X and θ displacements. The third group has no uncoupled Y modes, i.e., X, Y and θ displacements occur in each of the modes.

The difference in the first and the second groups is in the third, fourth and ninth modes while other modes are similar in form. Each of these modes in the first group differs from the corresponding modes in the second group in the displacements of the masses. In the third group no two cases show corresponding modes which display similar displacement patterns.

All cases show that relative displacements in the mode shapes are of comparable size. None of the cases considered showed any distinct changes in the displacements of the masses. In the two mass as well as the three mass systems it was noticed that the cases having similar combinations of masses and isolators showed similar displacements of the masses.

D. Amplitude Comparison Under Forced Excitation

The sinusoidal forcing functions in the X and Y directions are considered separately for each case. This has primarily been selected to be typical of a vibration testing apparatus, i.e., vertical and horizontal testing are usually done separately. As the systems are linear, the solution obtained by considering the forces together will be the same as superimposing solutions obtained by considering the forces separately.

Distances were considered above the c.g. of top mass where there is a possibility for external forces to be applied on the equipment placed on the systems. Sometimes the unbalanced forces in the equipment act horizontally at the base where they are attached to the mass and sometimes the net forces act through a point above the base of the equipment. In the vertical direction, the equipment may have forces which act through the c.g. of the top mass or to either side of the c.g. Hence, to consider typical possibilities where the external force may act, forces in the X direction were chosen at twelve and eighteen inches above the c.g. of the top masses. In the Y direction forces through the c.g. of the top mass and at eighteen inches left of the c.g. were considered. No external moments were considered but the moments created by the above mentioned forces about the c.g. of the top mass were considered.

All eighteen cases (see schematic representations in Appendix B) are examined for the amplitudes of the top mass with the forces in the X and the Y directions, separately. The main emphasis was placed on cases 1, 7, 8, 10 and 11 for they represent the largest and the smallest frequency bandwidths of the systems considered. However, the remaining cases were examined to see if any had amplitudes of the top mass higher or lower than the five selected cases.

The amplitudes of the top mass for each case were obtained for forcing frequencies, varying from three to fifteen cycles per second. All natural frequencies of the cases examined fall below three cps. Hence, to avoid any resonant conditions, three cps was chosen as a realistic starting forcing frequency to obtain amplitudes of the top mass. Also most commercial vibration shakers have their lowest frequencies near this value. Amplitudes were obtained up to fifteen cps only, as the pattern in the behavior of the amplitudes up to that forcing frequency is sufficient in observing the asympototic behavior for large frequencies.

Figures 4.7 and 4.8 show the absolute values of amplitudes X_1 and θ_1, respectively, of the top mass in the systems considered as a function of forcing frequency. These amplitudes are for the forcing function in the X direction at twelve inches above the c.g. Corresponding amplitudes for a similar force at eighteen inches above the c.g. are given in Figures 4.9 and 4.10. The Y amplitudes of the masses were negligibly small in these cases.

Figure 4.11 shows the absolute value of amplitude Y_1 of the top

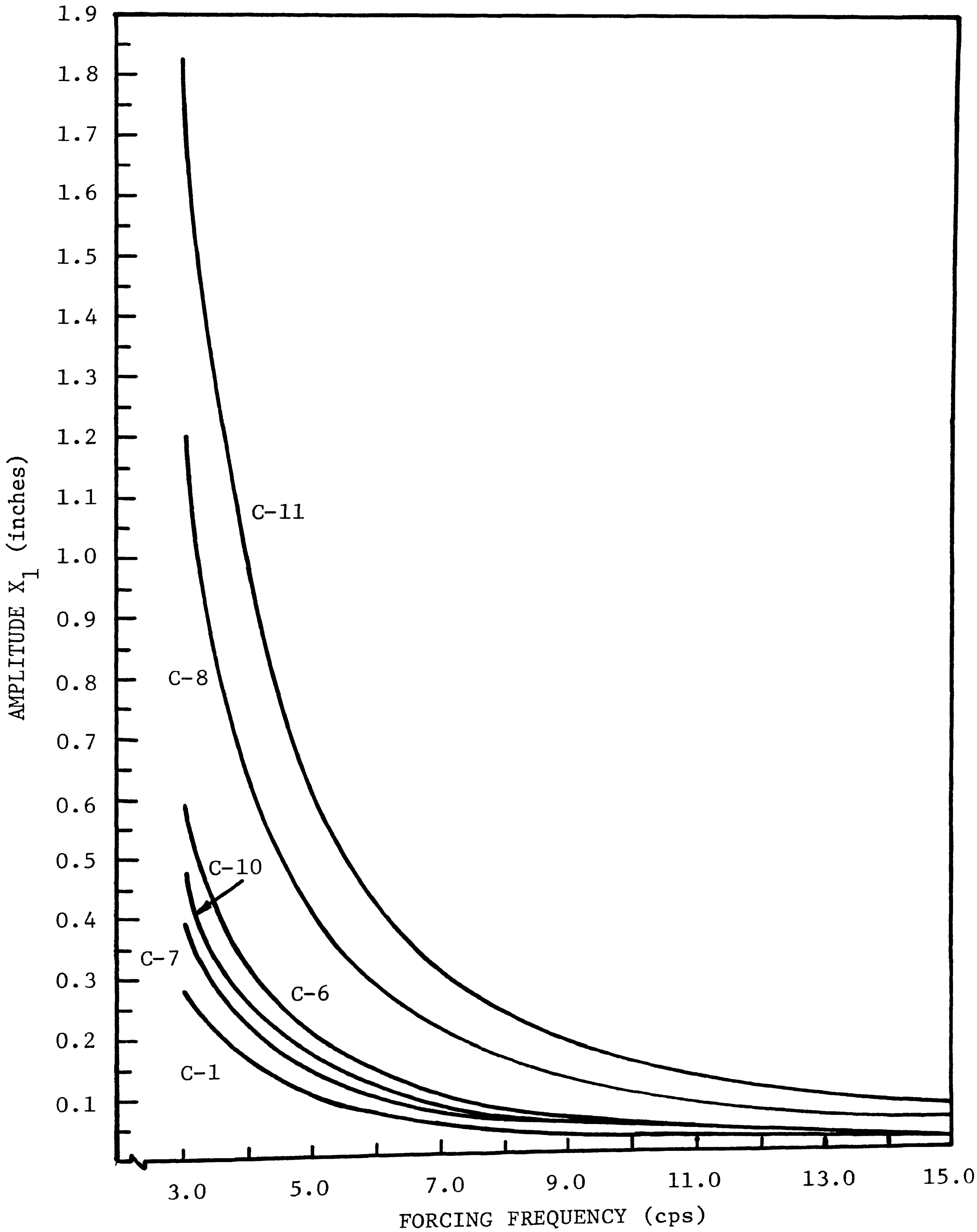


Fig. 4.7 Response (X_1) Curves for Forcing Function in the X-Direction at Twelve Inches Above c.g.

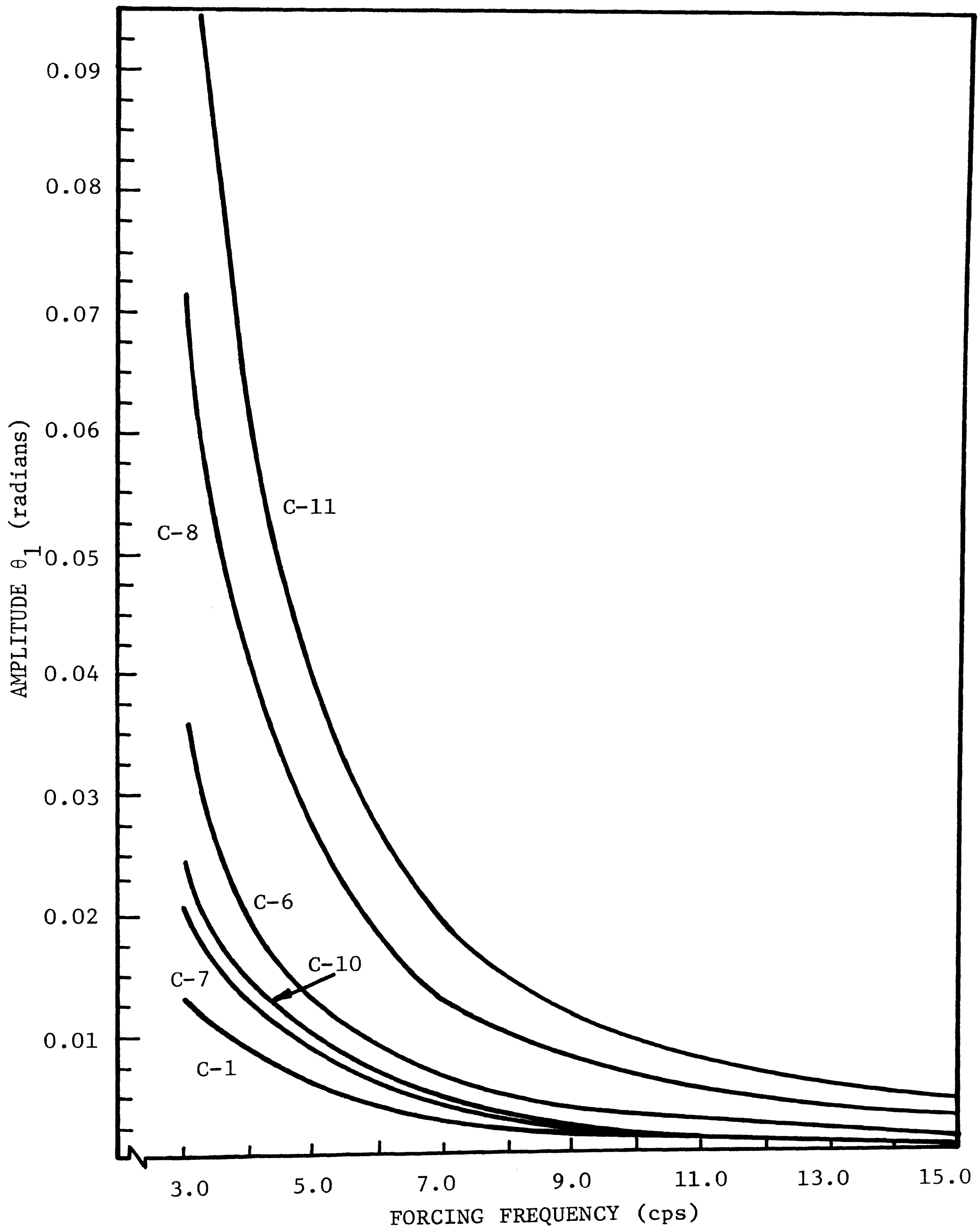


Fig. 4.8 Response (θ_1) Curves for Forcing Function in the X-Direction at Twelve Inches Above c.g.

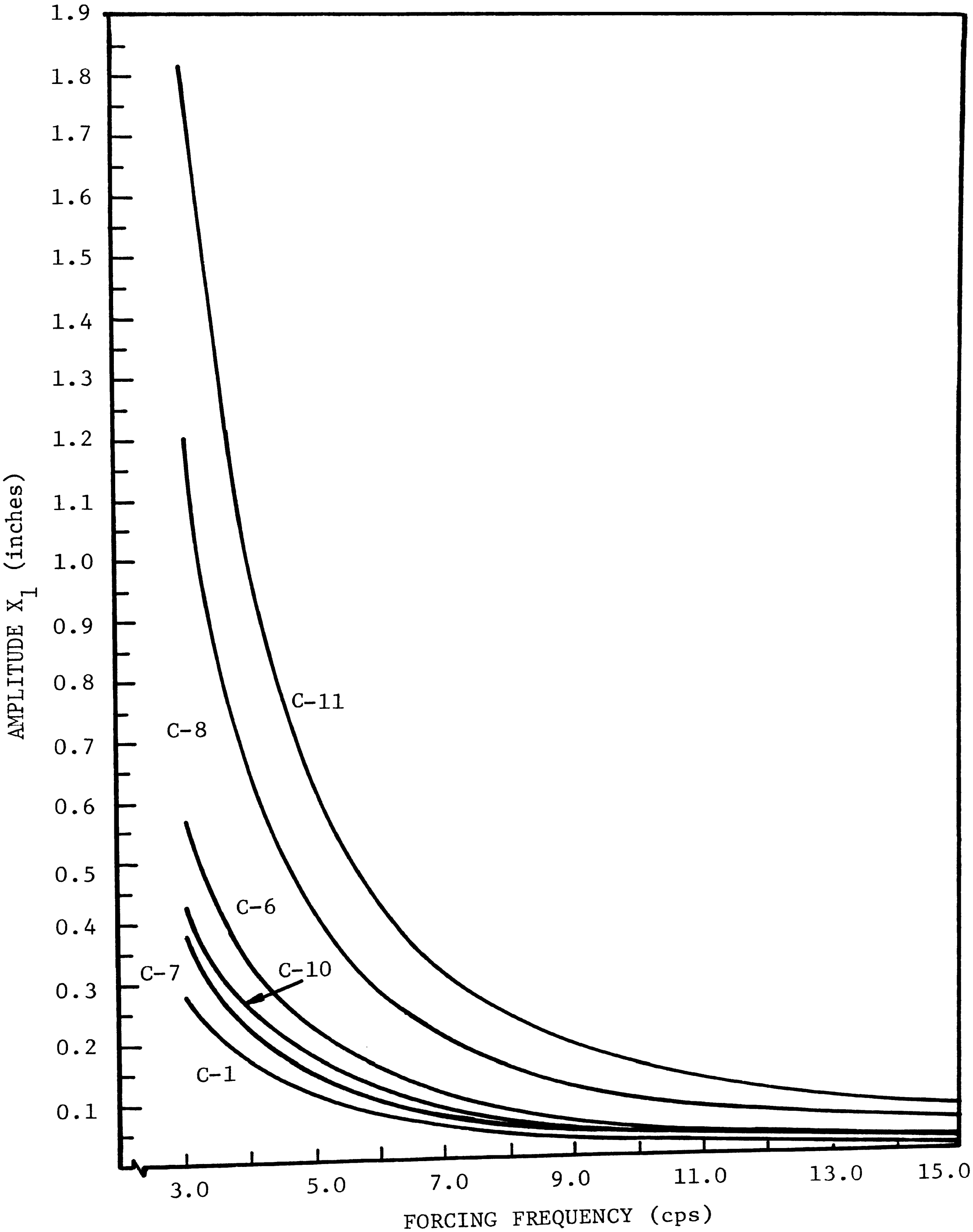


Fig. 4.9 Response (X_1) Curves for Forcing Function in the X-Direction at Eighteen Inches Above c.g.

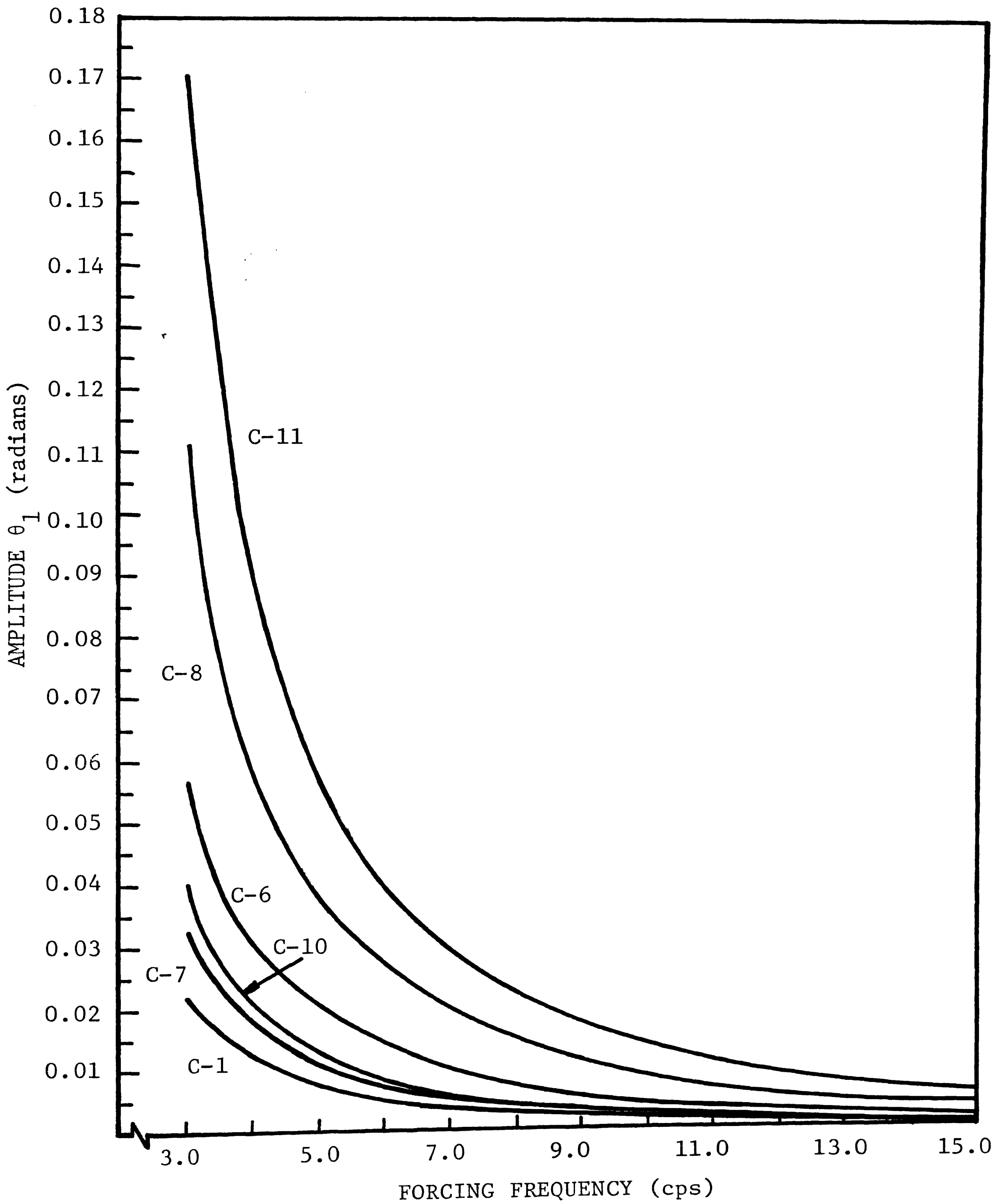


Fig. 4.10 Response (θ_1) Curves for Forcing Function in the X-Direction at Eighteen Inches Above c.g.

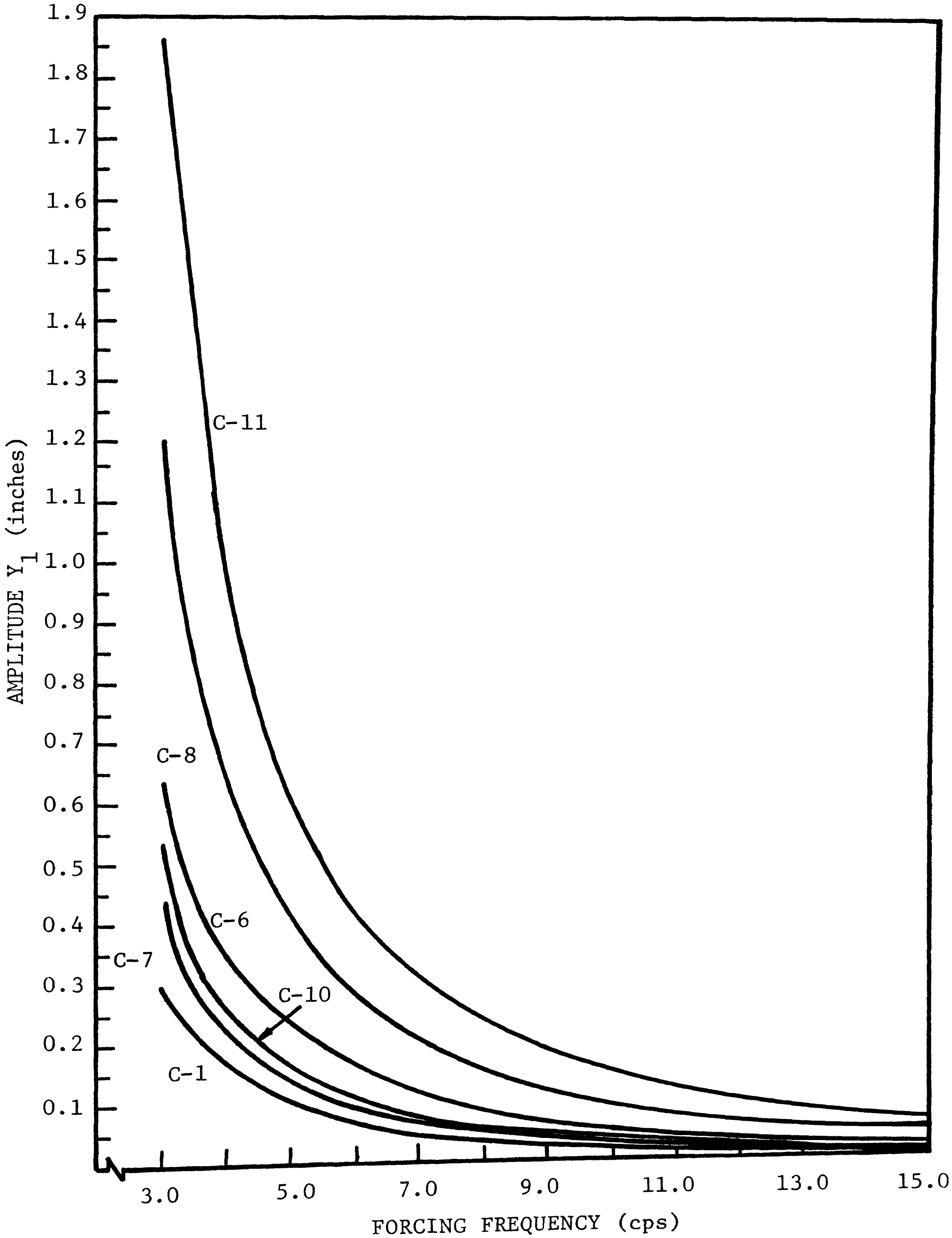


Fig. 4.11 Response (Y_1) Curves for Forcing Function in the Y-Direction Passing Through c.g.

mass as a function of forcing frequencies. These values are for the force in the Y direction through the c.g. Figures 4.12, 4.13 and 4.14 represent the absolute amplitudes X_1, Y_1 and θ_1, respectively, for the force in the Y direction at eighteen inches left of the c.g.

Values of all the amplitudes, except X_1 for a force in the Y direction at eighteen inches left of the c.g., increase according to the order of the cases 1, 7, 10, 6, 8 and 11, i.e., case 1 attains the lowest amplitudes and case 11 has the highest amplitudes. It can be seen that, for cases 1, 7 and 10 the amplitudes increase as the natural frequency bandwidth of these systems increase. But cases 6, 8 and 11 have smaller bandwidths than cases 7 and 10, and still show higher amplitudes than cases 7 and 10. This appears to be based upon the fact that cases 6, 8 and 11 either have equal mass or lighter mass at the top compared to the bottom mass.

The values of the amplitude X_1 for a force in the Y direction at eighteen inches left of the c.g. (Fig. 4.12) increase in the following order; cases 1, 8, 11, 7, 10 and 6. Here it may be noted that the amplitudes increase as the bandwidth increases except in case 6. In case 6 some isolators from M_1 go to the foundation directly. This amplitude X_1 differs from the other amplitudes and increases as the natural frequencies of the system increase. It is to be noted that the values of amplitudes X_1 for an eccentric force in the Y direction are very small compared to those of Y_1 amplitudes for the same force. In fact, the maximum values of Y_1 are about twenty times the maximum values of X_1.

Case 6 was not one of the main emphasized cases but is included

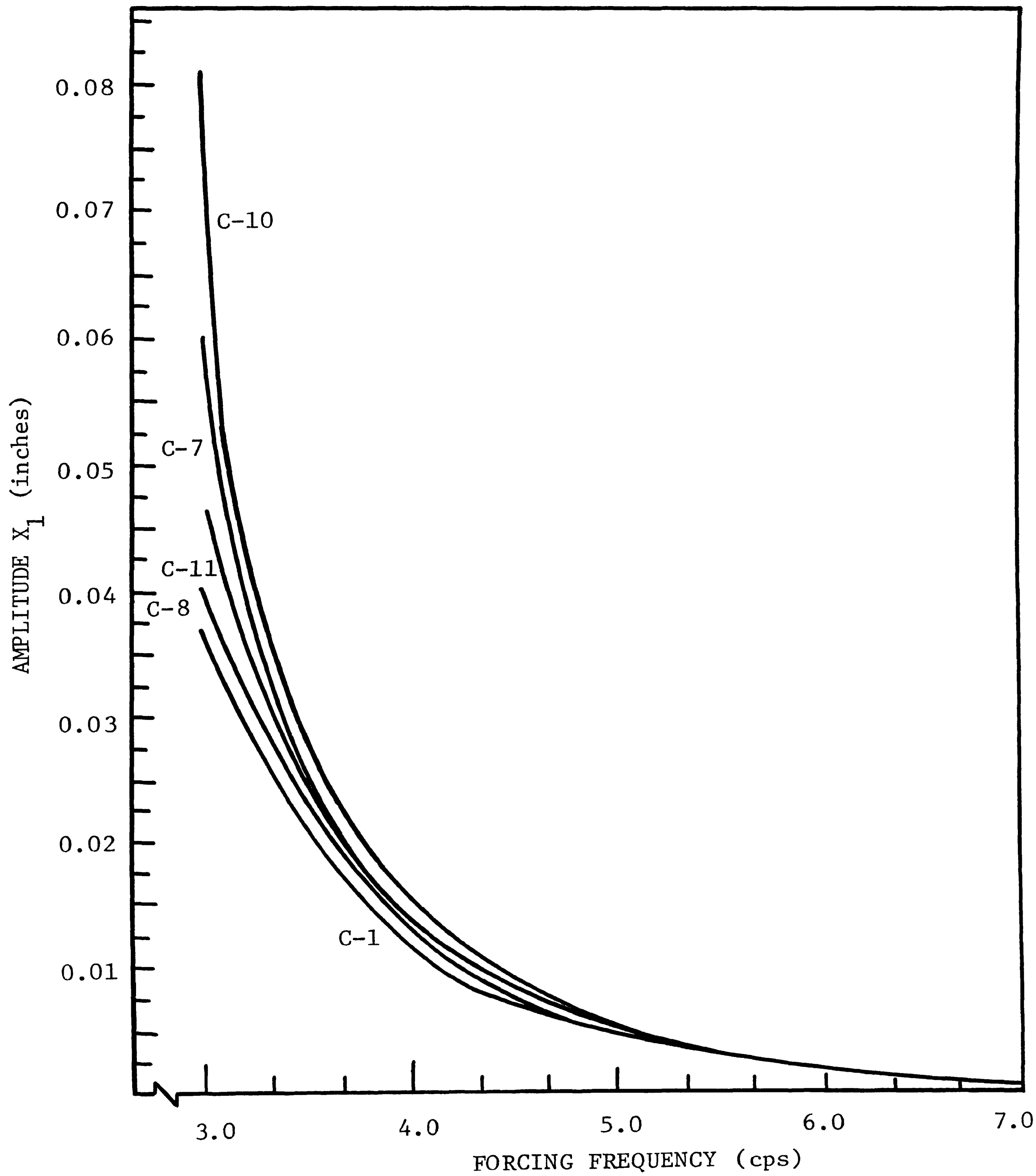


Fig. 4.12 Response (X_1) Curves for Forcing Function in the Y-Direction at Eighteen Inches Left of c.g.

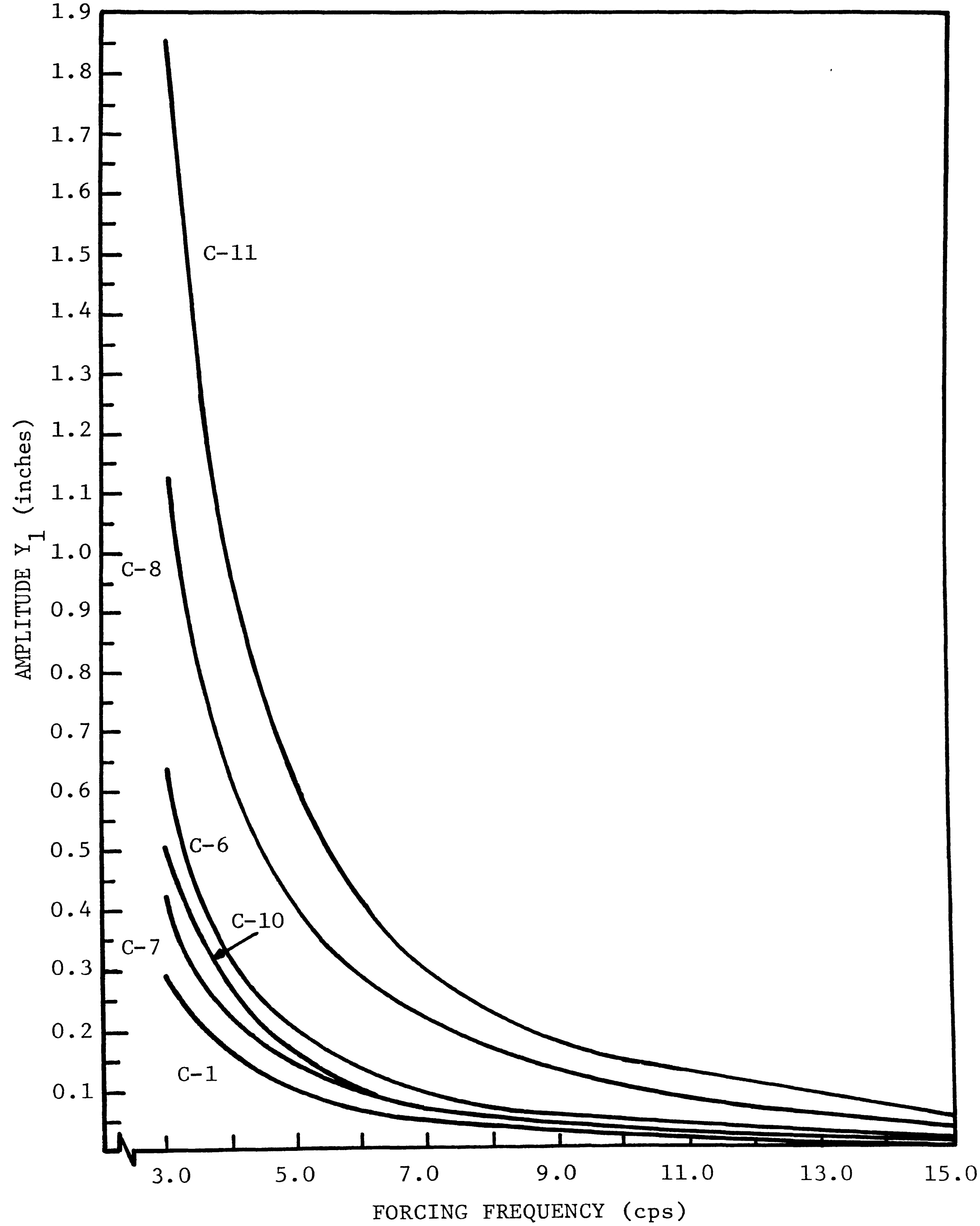


Fig. 4.13 Response (Y_1) Curves for a Forcing Function in the Y-Direction at Eighteen Inches Left of c.g.

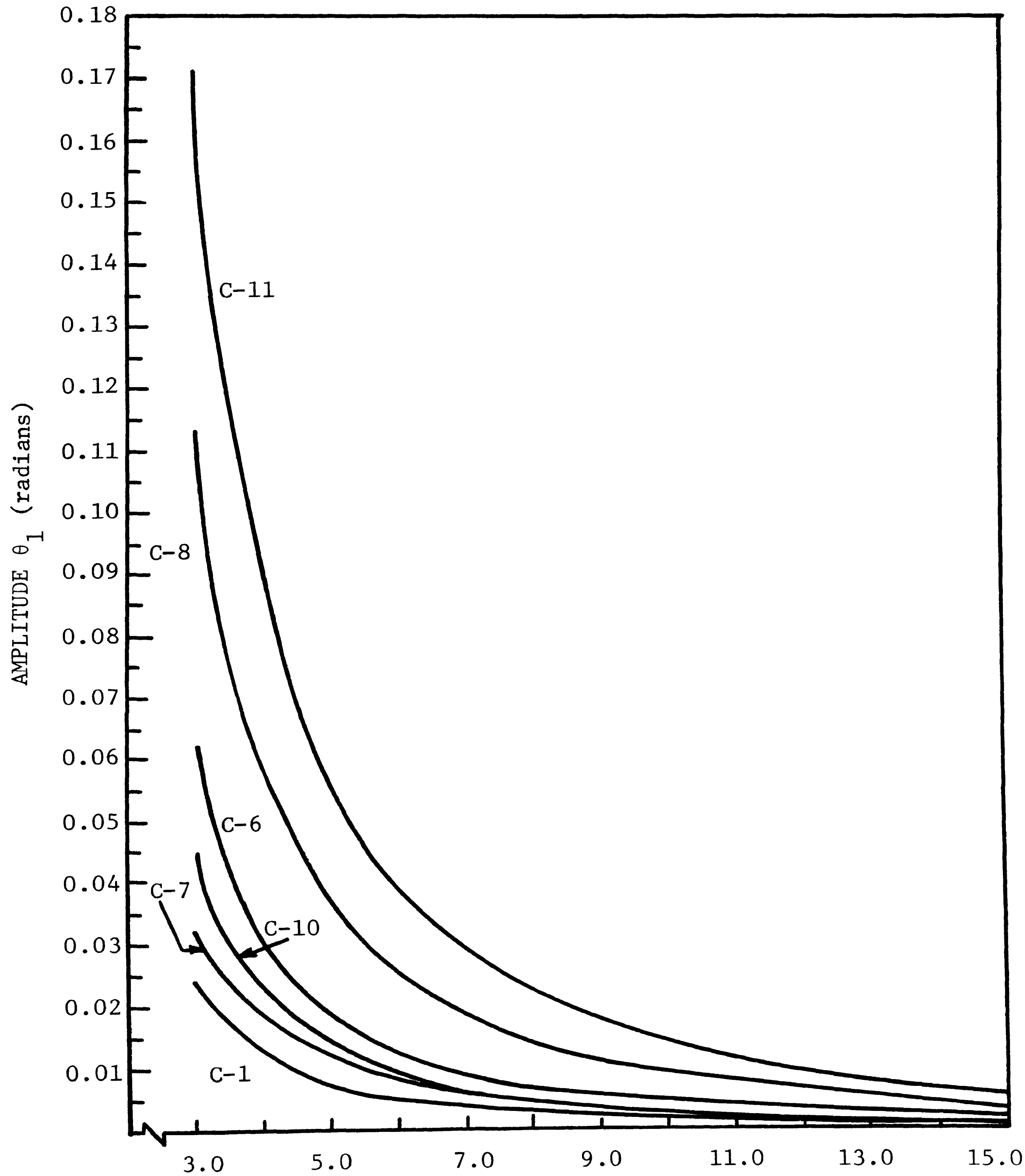


Fig. 4.14 Response (θ_1) Curves for Forcing Function in the Y-Direction at Eighteen Inches Left of c.g.

in the figures as being indicative of the remaining cases 3, 4, 5, 9, 12, 13, 14, 15, 16, 17 and 18. Amplitudes of these remaining cases lie between the maximum and minimum amplitudes of the five cases discussed earlier.

It can be concluded that amplitudes of the top mass of a system increase as the natural frequencies of the system increase, provided the heavier mass is located at the top. Systems having the smallest mass at the top have larger amplitudes than those having the heaviest mass at the top. The conventional one mass system has the lowest amplitudes. The maximum amplitudes of the cases having the largest amplitudes are about six times the maximum amplitudes of the cases having the smallest amplitudes.

E. Transmissibility Comparison

One purpose in providing a vibration isolator for a system is to attain a condition wherein the force or moment transmitted to the support is less than the force or the moment applied to the mass. Transmissibility indicates the attenuation of the force or moment being transmitted to the foundation.

The plot of force transmissibility in the X direction and moment transmissibility in the Z direction as a function of the forcing frequencies are shown in Figures 4.15 and 4.16, respectively, for the forcing function being in the X direction at twelve inches above c.g. The corresponding transmissibilities for the force in the X direction at eighteen inches above the c.g. are shown in Figures 4.17 and 4.18.

Figure 4.19 plots the force transmissibility in the Y direction for the force in the Y direction passing through the c.g. of the top

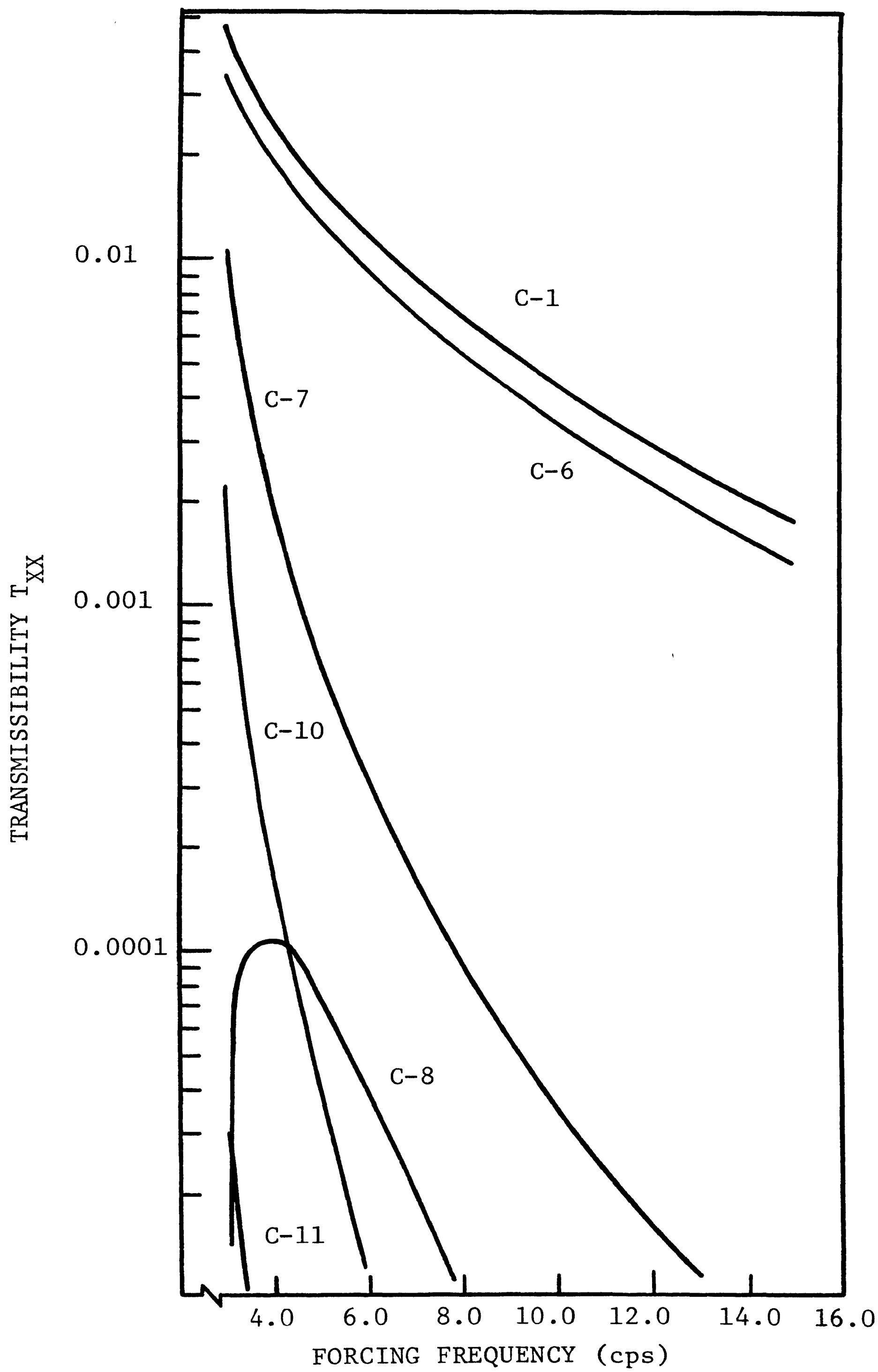


Fig. 4.15 Transmissibility (T_{XX}) Curves for a Forcing Function in the X-Direction at Twelve Inches Above c.g.

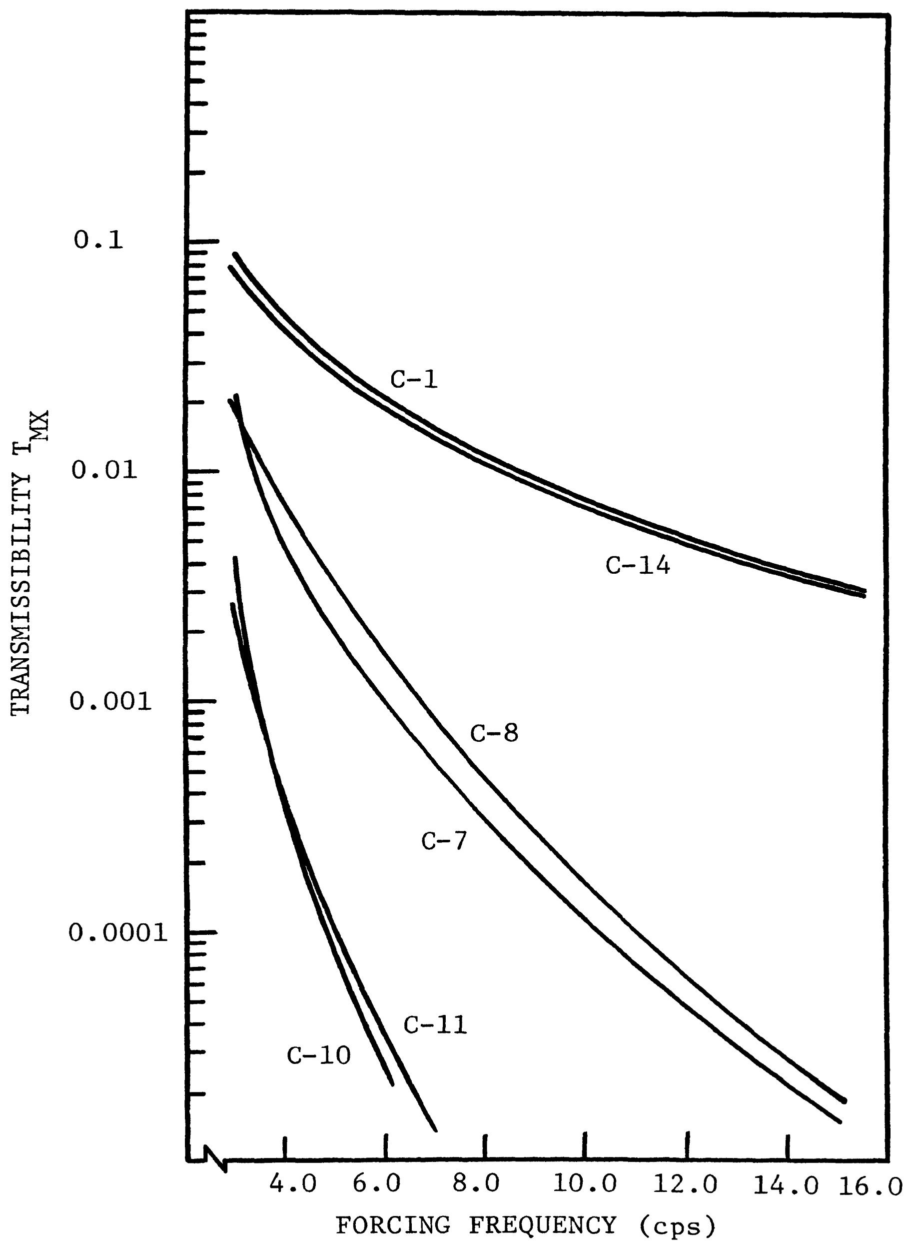


Fig. 4.16 Transmissibility (T_{MX}) Curves for a Forcing Function in the X-Direction at Twelve Inches Above c.g.

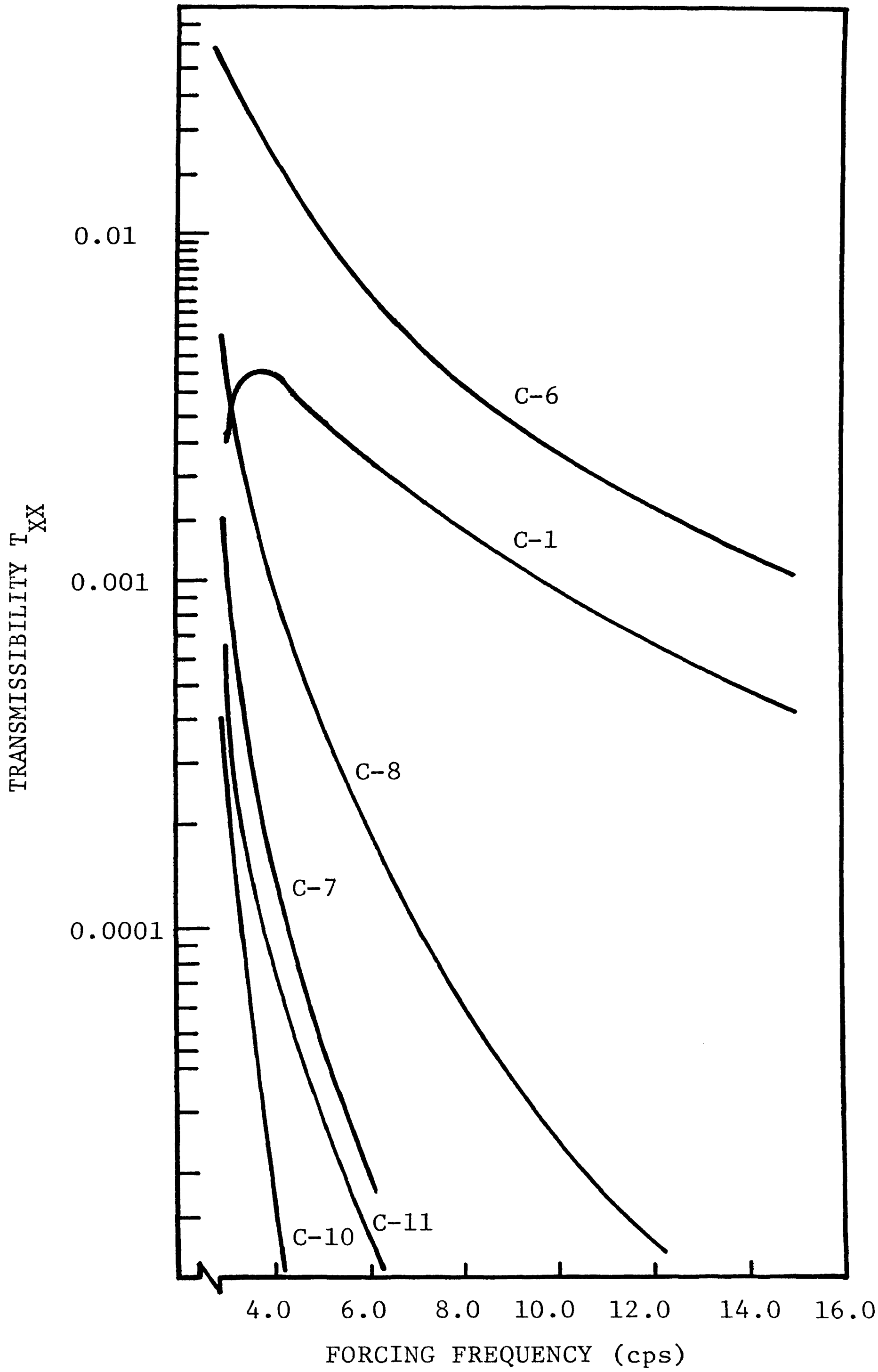


Fig. 4.17 Transmissibility (T_{XX}) Curves for a Forcing Function in the X-Direction at Eighteen Inches Above c.g.

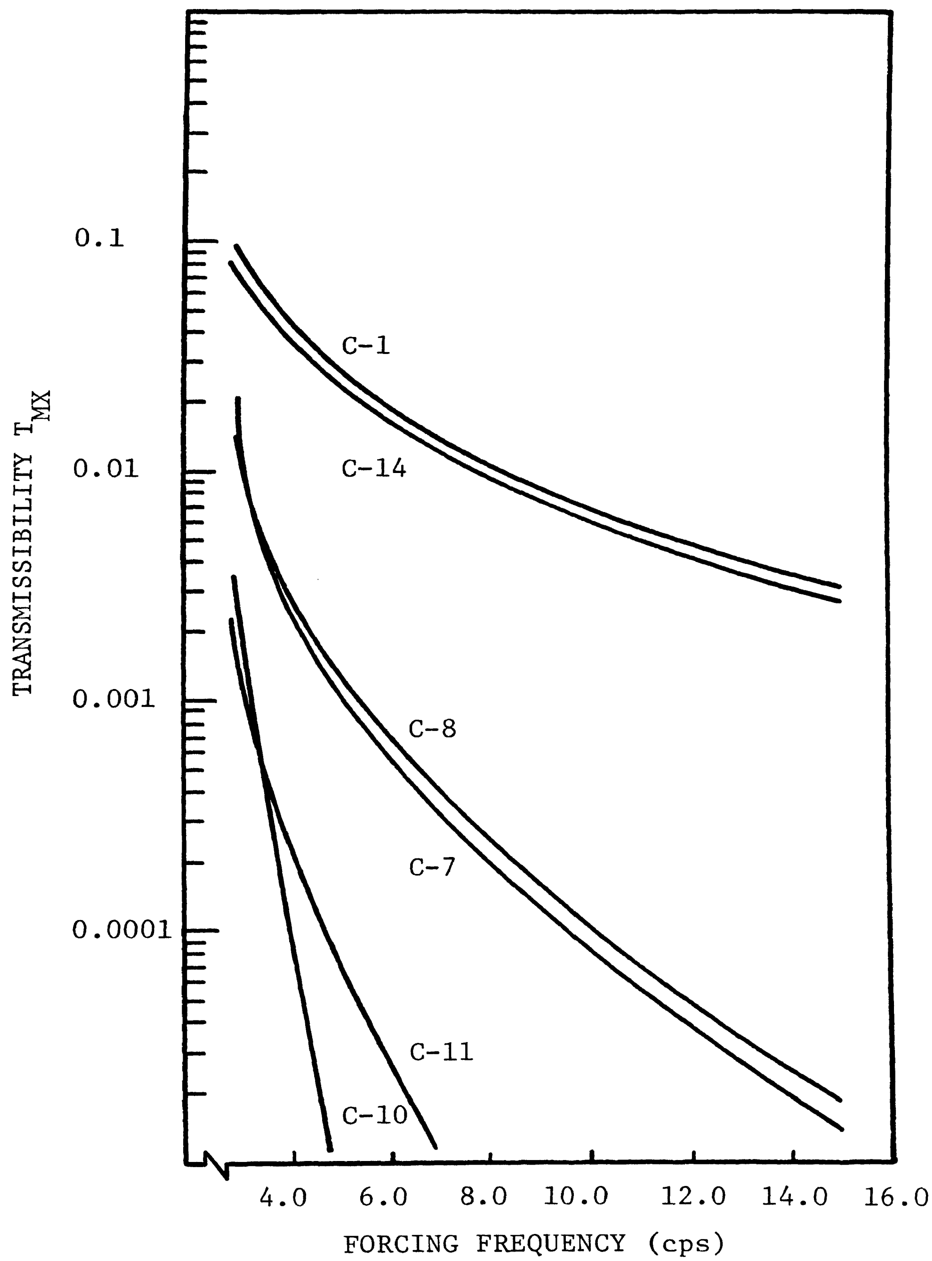


Fig. 4.18 Transmissibility (T_{MX}) Curves for a Forcing Function in the X-Direction at Eighteen Inches Above c.g.

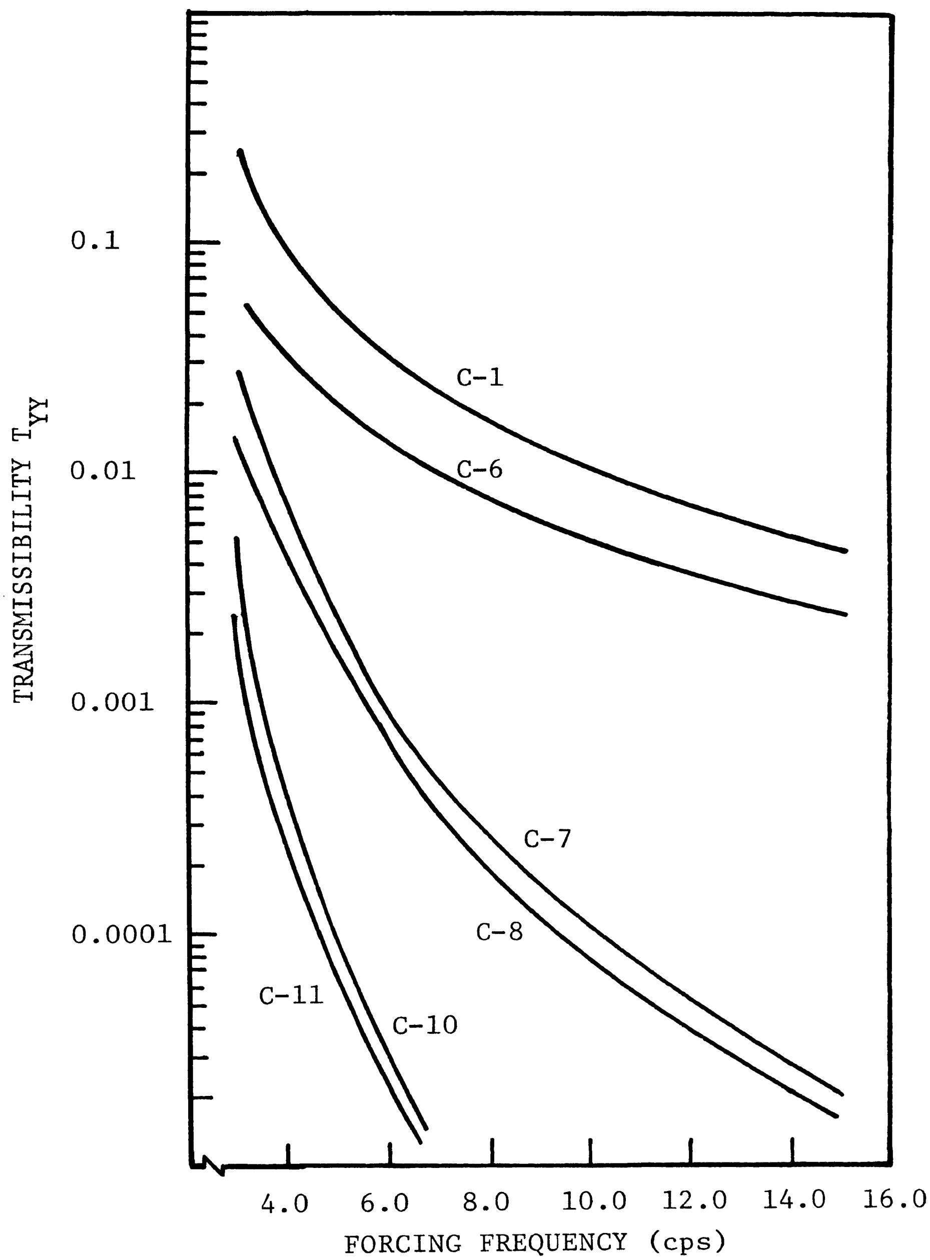


Fig. 4.19 Transmissibility (T_{YY}) Curves for a Forcing Function in the Y-Direction Through the c.g.

mass. For the force in the Y direction at eighteen inches left of the c.g., the force transmissibilities in the X and Y directions and moment transmissibility are represented by Figures 4.20, 4.21 and 4.22, respectively. These transmissibilities are considered as a function of forcing frequencies also.

As in the amplitude comparison, cases 1, 7, 8, 10 and 11 are emphasised for transmissibility considerations. The remaining cases have been examined for higher or lower transmissibilities than the five cases mentioned above. As case 6 was considered in the comparison of amplitudes, it is also included here to see how its transmissibility compares with the other five cases. In addition, case 14 is included for moment transmissibilities as it shows higher moment transmissibility than case 6 but lower than case 1.

Table I indicates by case number the order in which the transmissibilities decrease.

It is seen from the results that, the force and moment transmissibilities in the three mass systems (except cases 14 and 16) are lower than those for the two mass and conventional one mass systems.

Cases 14 and 16 differ from the typical three mass system in that they have some isolators going to the foundation directly from the top mass.

The conventional one mass system has the highest force and moment transmissibilities. It is higher than cases 14 and 16. An exception to this is the force transmissibility for the force in the X direction at eighteen inches above the c.g. of top mass. Case 6 has higher force transmissibility than the conventional one mass

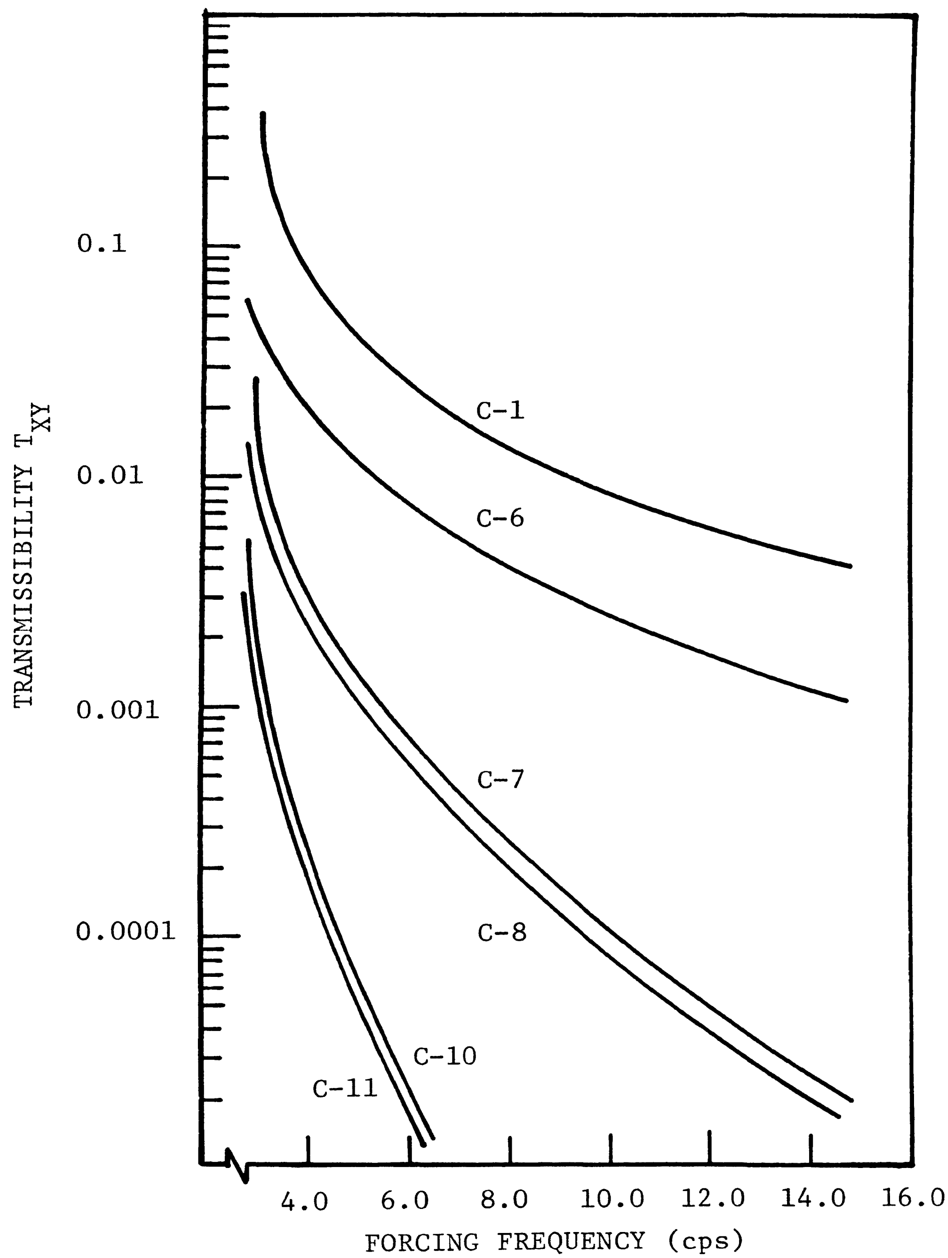


Fig. 4.20 Transmissibility (T_{XY}) Curves for a Forcing Function in the Y-Direction Eighteen Inches Left of c.g.

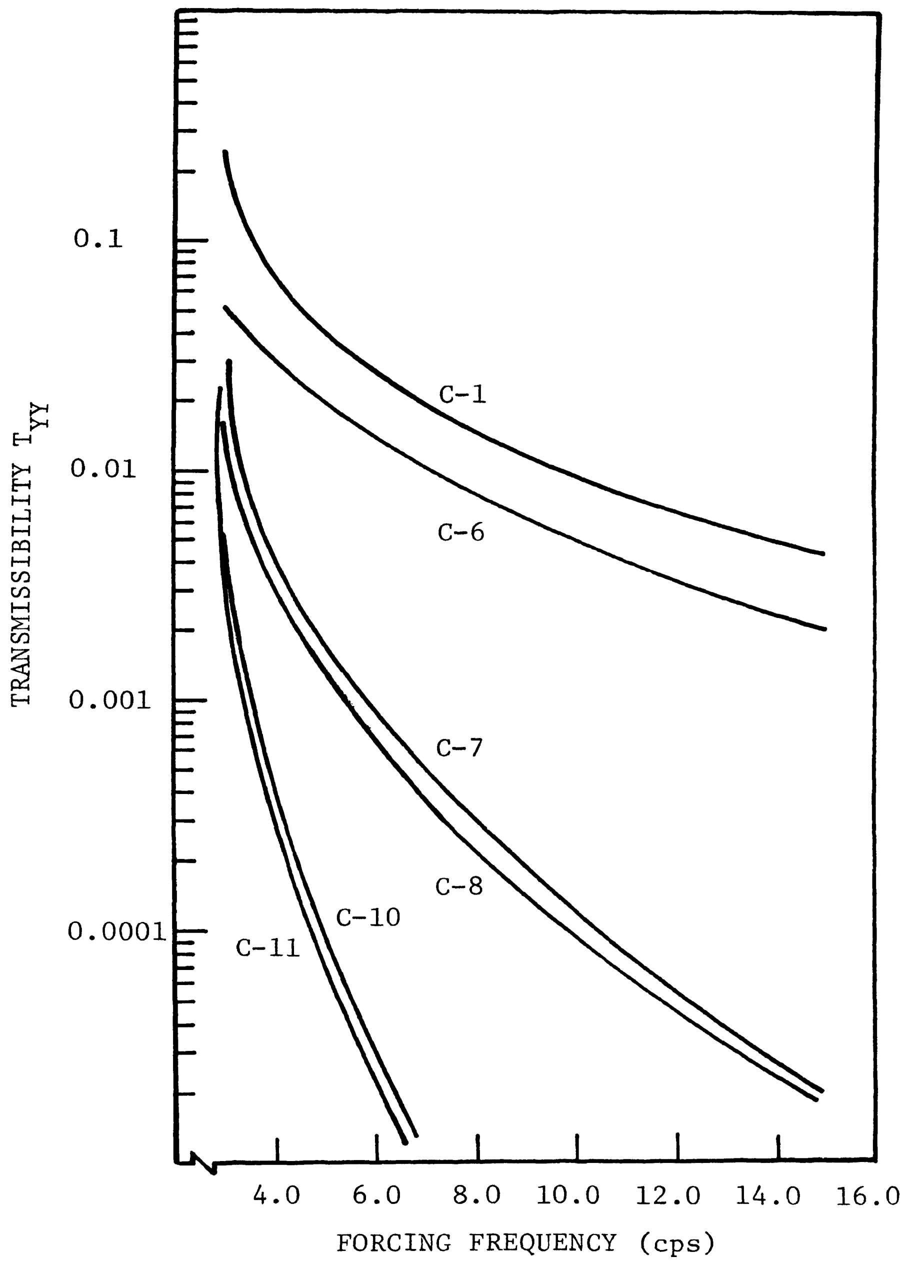


Fig. 4.21 Transmissibility (T_{YY}) Curves for a Forcing Function in the Y-Direction Eighteen Inches Left of c.g.

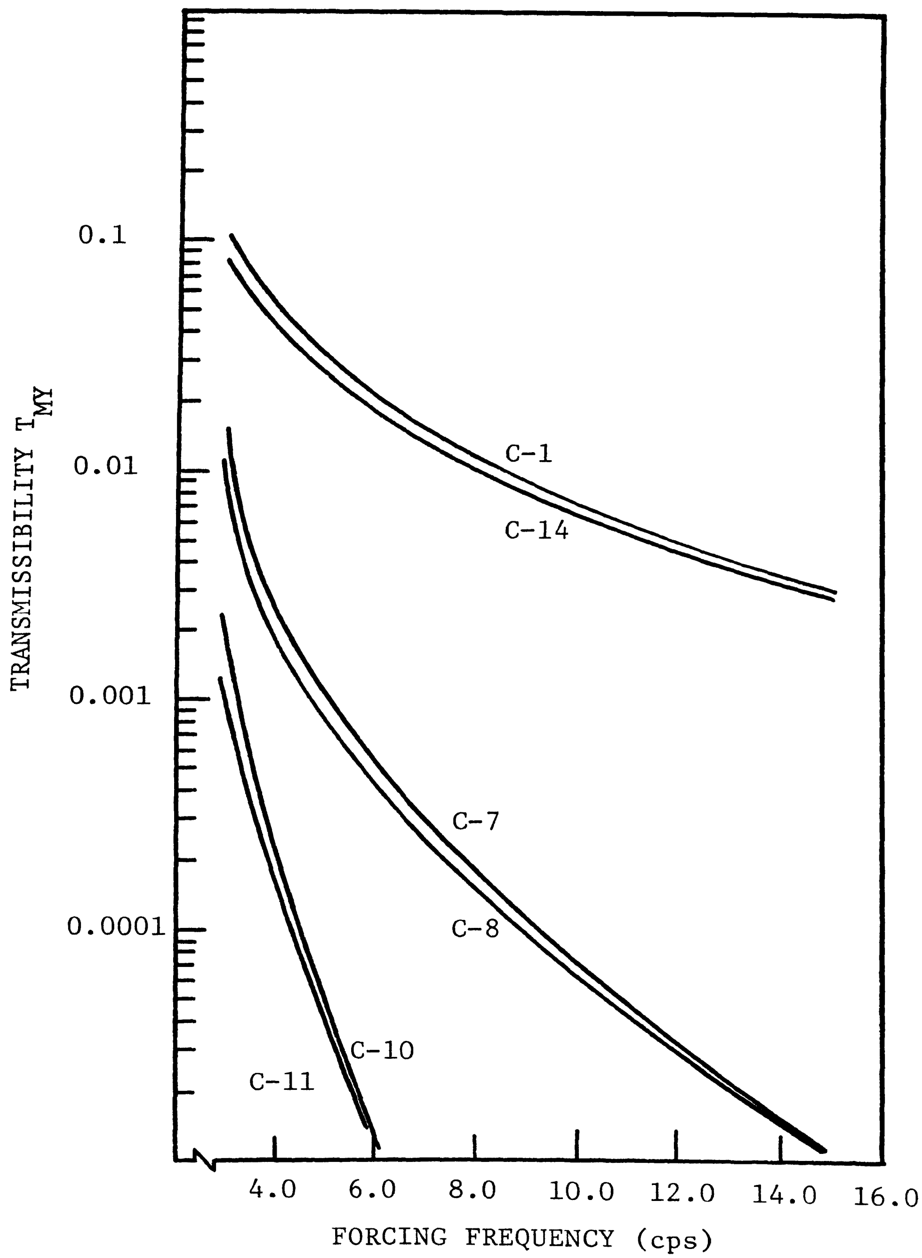


Fig. 4.22 Transmissibility (T_{MY}) Curves for a Forcing Function in the Y-Direction Eighteen Inches Left of c.g.

Table I

Case Numbers in Order of Decreasing Transmissibility

	Eccy=12.0"		Eccy=18.0"		Eccx=0.0"	Eccx=18.0"		
	T_{XX}	T_{MX}	T_{XX}	T_{MX}	T_{YY}	T_{XY}	T_{YY}	T_{MY}
H.T. Case no.	1	1	6	1	1	1	1	1
	6	14	1	14	6	6	6	14
	7	8	8	8	7	7	7	7
	10	7	7	7	8	8	8	8
	8	11	11	11	10	10	10	10
L.T. Case no.	11	10	10	10	11	11	11	11

where: H.T. = Highest transmissibility, and

L.T. = Lowest transmissibility.

system.

In most cases the maximum transmissibility of the highest transmissibility case is about fifty times more than the maximum transmissibility of the lowest transmissibility case.

It has been concluded that, for minimum amplitudes a cascade system with the heavier mass at the top or the conventional one mass system, are better cases. For minimum transmissibility the cases of the three mass system (except 14 and 16) are the best.

Upon evaluation of all the figures, it appears possible to choose a more optimum case considering both amplitude response and transmissibility. Case 10 appears plausible as its maximum amplitudes are nearly as low as any of the cases considered and lower than most. In addition, case 10 has a transmissibility which is much lower than most of the cases, being only slightly higher than the transmissibility for case 11. Case 10 does increase amplitude somewhat over the classical one mass system, but it also decreases or improves transmissibility over the one mass system by a factor of approximately one to two orders of magnitude.

CHAPTER V

CONCLUSIONS

Having examined and compared eighteen cascaded systems, several conclusions can be reached about their isolation properties.

1. Case 1 which represents the classical one mass system has the lowest natural frequency bandwidth while case 10, which is the adjacently connected three mass system, has the highest.

2. Relative displacements in the principal mode shapes are of comparable size. No distinct changes in the mode shape displacements are observed. Systems having similar mass and isolator combinations display similar displacement patterns.

3. Absolute values of amplitudes of the top mass increase with the natural frequency bandwidth for cases having heavier masses at the top. The cases having lighter masses at the top have larger amplitudes than those with the heavier masses at the top. The conventional one mass system has the lowest amplitudes by a factor of approximately two in comparison to the best cascaded system.

4. Low force and moment transmissibilities are obtained in the three mass system except those cases in which springs connect directly to the foundation from the top mass. The conventional one mass system has the largest transmissibility.

5. A more optimum system considering both amplitude response and transmissibility is represented by case 10. This system increases response somewhat over the classical one mass system but improves transmissibility by a factor of nearly 50. In addition, case 10 has the widest natural frequency bandwidth.

APPENDIX A

ILLUSTRATIVE DETAILS OF A CASCADE SYSTEM

ILLUSTRATIVE DETAILS OF A CASCADE SYSTEM

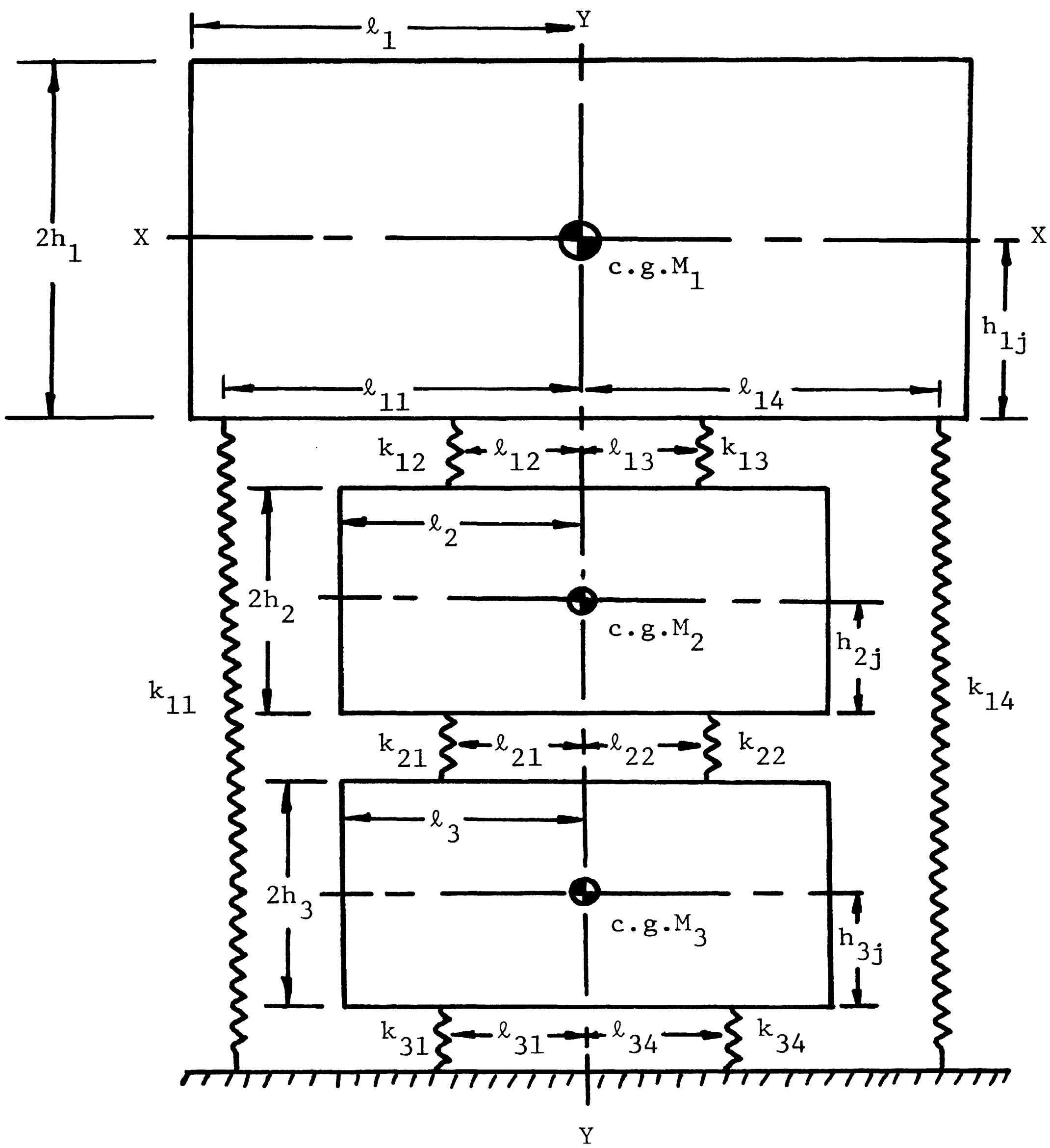


Fig. A.1 Details of Parameters Listed in Tables II and III

Figure A.1 shows the parameters used in tables II and III. Units of the parameters are:

$$M = \text{lbs. sec}^2\text{./in.}$$

$$I_i = \text{lbs. sec}^2\text{. in.}$$

$$\ell_{ij} = \text{inches}$$

$$h_{ij} = \text{inches}$$

$$k_{ij} = \text{lbs./in.}$$

Details of masses and isolators for all the cases are shown in tables II and III.

Table II

DETAILS OF MASSES

One Mass System

Case No.	M_1	I_1	$2\ell_1$	$2h_1$	j
1*	10.36	2490.	48.	24.	4
2*	10.36	2490.	48.	24.	4

Two Mass System

Case No.	M_1	I_1	M_2	I_2	$2\ell_1$	$2h_1$	$2\ell_2$	$2h_2$	j_{M1}	j_{M2}
3**	6.90	1473.8	3.45	681.6	48.	16.	48.	8.	4	4
4**	3.45	681.6	6.90	1473.8	48.	16.	48.	16.	4	4
5**	5.18	1057.0	5.18	1057.0	48.	12.	48.	12.	4	4
6+	5.18	1057.0	5.18	670.1	48.	12.	36.	16.	6	4
7**	7.77	1702.0	2.59	505.2	48.	18.	48.	6.	4	4
8**	2.59	505.2	7.77	1702.0	48.	6.	48.	18.	4	4
9+	7.77	1702.0	2.59	293.6	48.	18.	36.	8.	6	4

* Figure 2.2, ** Figure 2.6, + Figure 2.10

Three Mass System

Case No.	M_1	I_1	M_2	I_2	M_3	I_3	$2\ell_1$	$2h_1$	$2\ell_2$	$2h_2$	$2\ell_3$	$2h_3$	j_{M1}	j_{M2}	j_{M3}
10*	6.90	1473.8	1.73	333.9	1.73	333.9	48.	16.	48.	4.	48.	4.	4	4	4
11*	1.73	333.9	1.73	333.9	6.90	1473.8	48.	4.	48.	4.	48.	16.	4	4	4
12*	5.18	1057.0	2.59	505.2	2.59	505.2	48.	12.	48.	6.	48.	6.	4	4	4
13*	2.59	505.2	2.59	505.2	5.18	1057.0	48.	6.	48.	6.	48.	12.	4	4	4
14*	5.18	1057.0	2.59	293.6	2.59	293.6	48.	12.	36.	8.	36.	8.	6	4	4
15	5.18	670.1	2.59	505.2	2.59	293.6	36.	16.	48.	6.	36.	8.	4	6	4
16+	5.18	1057.0	2.59	293.6	2.59	155.4	48.	12.	36.	8.	24.	12.	6	6	4
17**	5.18	1057.0	2.59	293.6	2.59	505.2	48.	12.	36.	8.	48.	6.	6	4	4
18**	2.59	505.0	2.59	293.6	5.18	1057.0	48.	6.	36.	8.	48.	12.	6	4	4

* Figure 2.9

** Figure 2.11

\+ Figure A.1

Table III

DETAILS OF ISOLATORS

One Mass System

Case No.	k_{1j}	ℓ_{1j}	h_{1j}
1	102*	18,6,6,18	12*
2	102.	18,6,6,18	12.

Two Mass System

Case No.	K_{1j}	ℓ_{1j}	h_{1j}	k_{2j}	ℓ_{2j}	h_{2j}
3	68.0*	18,6,6,18	8*	34.0*	18,6,6,18	4*
4	34.0	18,6,6,18	4.	68.0	18,6,6,18	8.
5	51.0	18,6,6,18	6.	51.0	18,6,6,18	6.
6	34.0	21,13.5,4.5,4.5,13.5,21	6.	51.0	13.5,4.5,4.5,13.5	8.
7	76.5	18,6,6,18	9.	25.5	18,6,6,18	3.
8	25.5	18,6,6,18	3.	76.5	18,6,6,18	9.
9	51.0	21,13.5,4.5,4.5,13.5,21	9.	25.5	13.5,4.5,4.5,13.5	4.

* Values valid for all j in k_{1j}, k_{2j}, h_{1j} and h_{2j} in all cases and j = 1,2,3,..., no. of isolators (as mentioned in table II)

Three Mass System

Case No.	k_{1j}	ℓ_{1j}	h_{1j}	k_{2j}	ℓ_{2j}	h_{2j}	k_{3j}	ℓ_{3j}	h_{3j}
10	68.0*	18,6,6,18	8*	17.0*	18,6,6,18	2*	17.0*	18,6,6,18	2*
11	17.0	18,6,6,18	4.	17.0	18,6,6,18	2.	68.0	18,6,6,18	8.
12	51.0	18,6,6,18	6.	25.5	18,6,6,18	3.	25.5	18,6,6,18	3.
13	25.5	18,6,6,18	3.	25.5	18,6,6,18	3.	51.0	18,6,6,18	6.
14	34.0	21,13.5,4.5,4.5,13.5,21	6.	25.5	13.5,4.5,4.5,13.5	4.	25.5	13.5,4.5,4.5,13.5	4.
15	51.0	13.5,4.5,4.5,13.5	8.	17.0	21,13.5,4.5,4.5,13.5,21	3.	25.5	13.5,4.5,4.5,13.5	4.
16	34.0	21,13.5,4.5,4.5,13.5,21	6.	17.0	16,8,4,4,8,16	4.	25.5	8,4,4,8	6.
17	34.0	21,13.5,4.5,4.5,13.5,21	6.	25.5	13.5,4.5,4.5,13.5	4.	25.5	13.5,4.5,4.5,13.5	3.
18	25.5	21,13.5,4.5,4.5,13.5,21	3.	25.5	13.5,4.5,4.5,13.5	4.	51.0	13.5,4.5,4.5,13.5	6.

* Values valid for all j in k_{1j}, k_{2j}, k_{3j}, h_{1j}, h_{2j} and h_{3j} for all the cases and j = 1,2,..., no. of isolators (as mentioned in table II)

APPENDIX B

SCHEMATIC REPRESENTATION OF CASCADE SYSTEMS

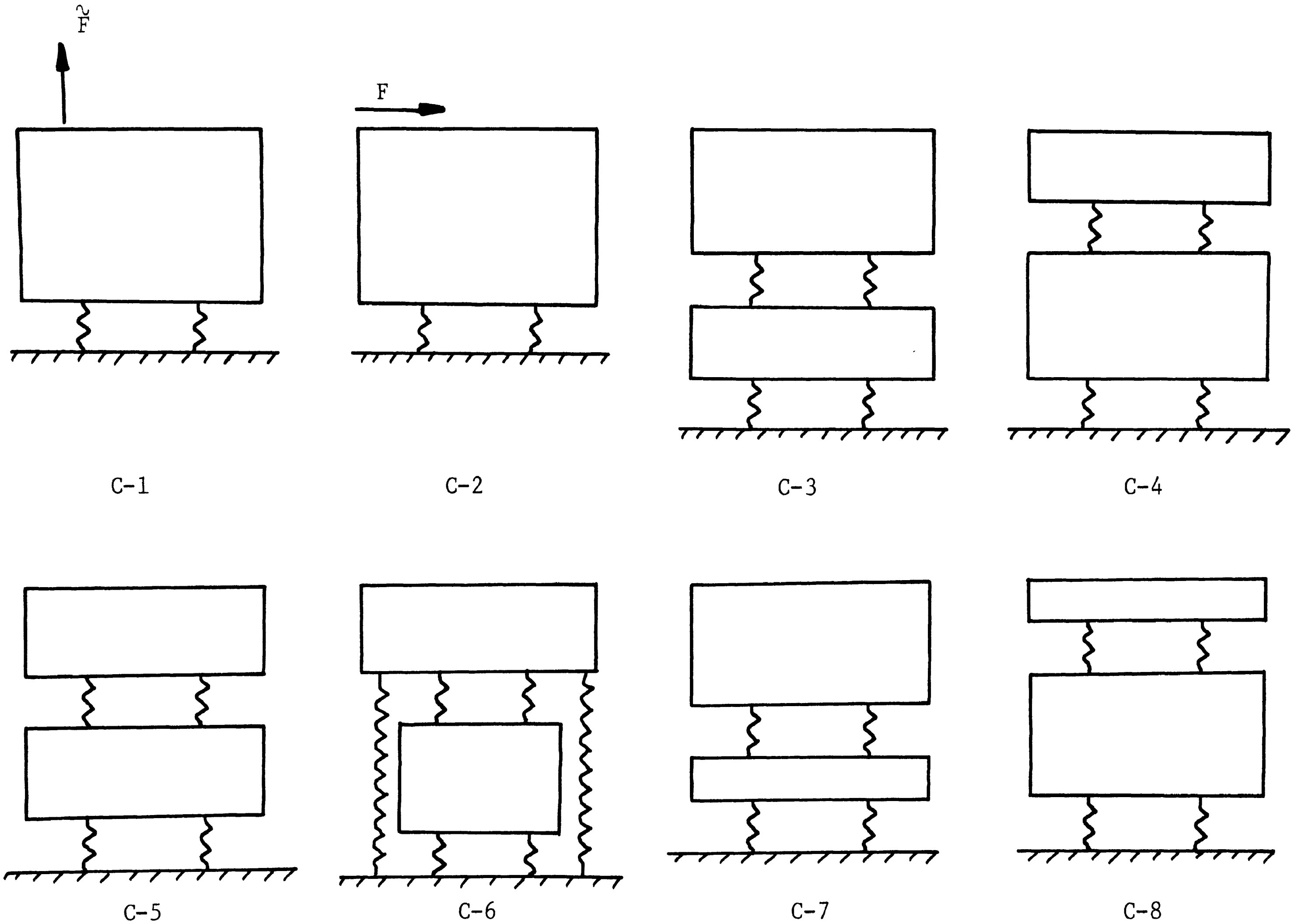


Fig. B.1 Schematic Representation of the One and Two Mass Systems

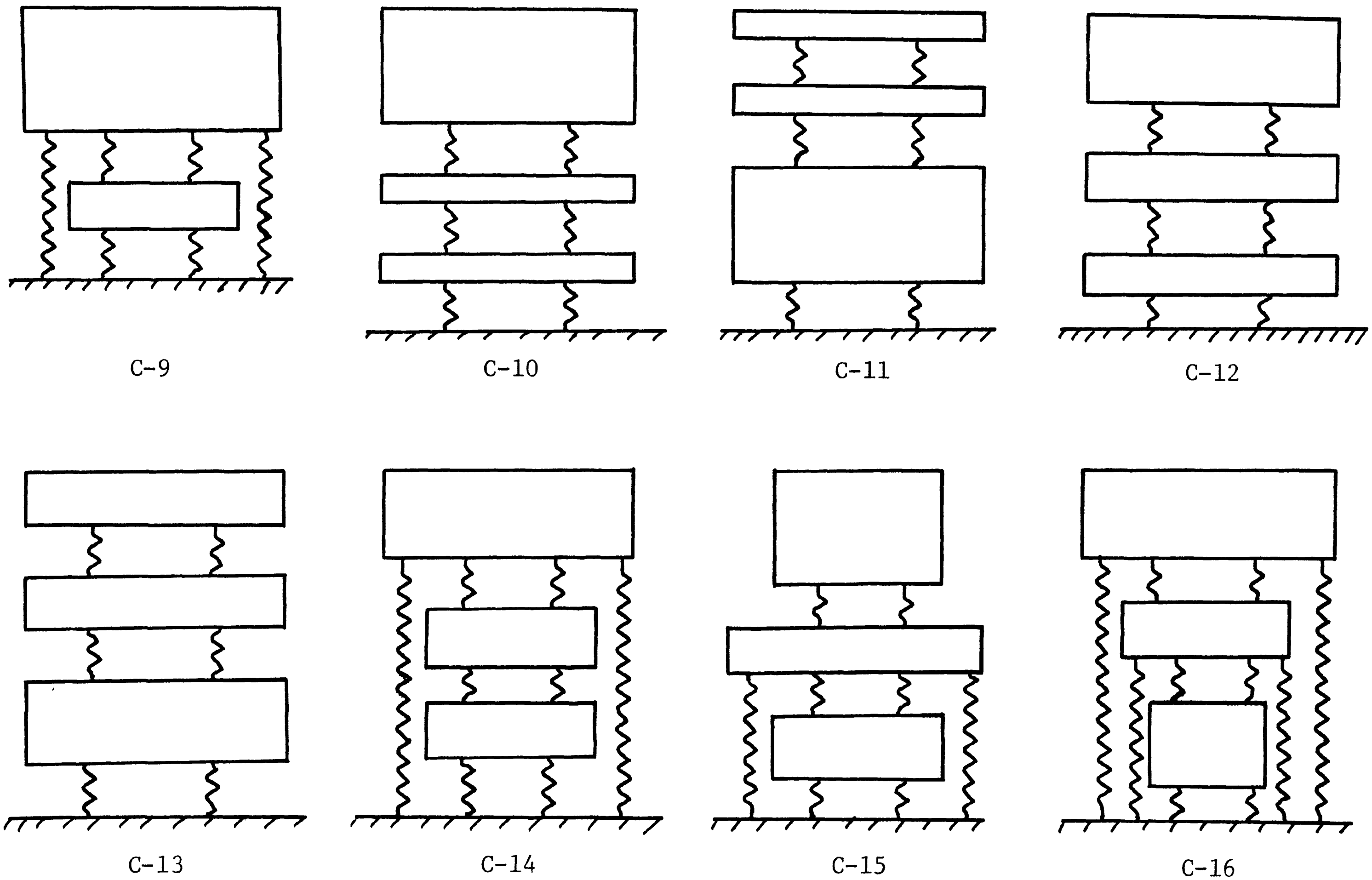


Fig. B.2 Schematic Representation of the Two and Three Mass Systems

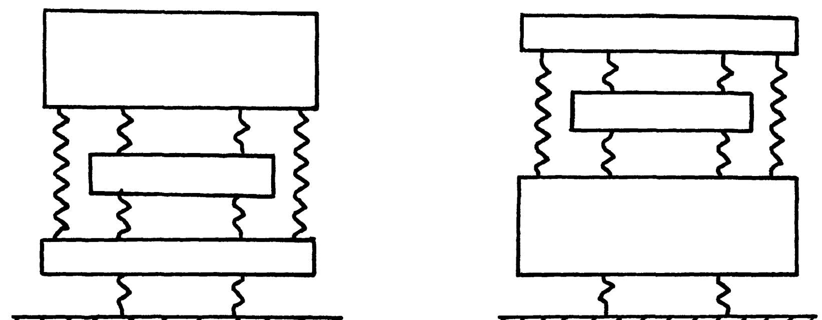


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Fig. B.3 Schematic Representation of the Three Mass System

CHAPTER VI

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CHAPTER VII

VITA

Rajnikant Bhikhabhai Patel was born on February 5, 1945 in Maragua, Kenya (East Africa). He received his school education from Duke of Gloucester School, Nairobi, Kenya. He received his science education from Bombay, India and a Bachelor of Engineering degree in Mechanical Engineering from Sardar Patel University in Anand, India. He was married to Miss Tarulatta N. Amin in December, 1968.

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